A two-component variant of the famous result on equilibration in scalar Fokker-Planck equations

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Diffusion, systems, convergence (and our humble contribution)

In the beginning...

The non-linear Fokker-Planck equation

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$$\bar{\rho}(x) = \left(A - B|x|^2\right)_+^{1/(m-1)}$$

and the Lyapunov functional

$$\mathbf{L}(\rho) = \int_{\mathbb{R}^d} \left[\frac{\rho(x)^m}{m-1} + \frac{\lambda}{2}|x|^2\rho(x)\right] \mathrm{d}x - L.$$

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Theorem ([Otto'01, Carrillo&Toscani'00, Dolbeault&DelPino'02])

The entropy is dissipated exponentially fast,

$$\mathbf{L}(\rho(t)) \leq \mathbf{L}(\rho(0)) \exp(-2\lambda t),$$

and consequently,

$$\left\|\rho(t) - \bar{\rho}\right\|_{L^1(\mathbb{R}^d)} \leq C \exp(-\lambda t).$$

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Later: Zillons of generalizations, e.g. [AGS'05,CJMTU'01]

- $\rho^m \rightsquigarrow f(\rho)$ subject to McCann's condition;
- $\lambda x \rightsquigarrow \nabla V(x)$ with $\nabla^2 V \geq \lambda \mathbf{1}$;
- add $\nabla \cdot (\rho \nabla W * \rho)$ with $\nabla^2 W \geq 0$.

Scalar to systems

What about coupled systems?

$$\partial_t \rho_1 = \Delta(\rho_1^m) + \nabla \cdot (\rho_1 \nabla V_1) + \Delta h_1(\rho_1, \rho_2) + g_1(\rho_1, \rho_2) + \cdots ,$$
$$\partial_t \rho_2 = \Delta(\rho_2^m) + \nabla \cdot (\rho_2 \nabla V_2) + \Delta h_1(\rho_1, \rho_2) + g_2(\rho_1, \rho_2) + \cdots .$$

There are numerous results on exponential convergence to …

- … homogeneous steady states, for diagonal diffusion
  e.g. [Desvillettes&Fellner'06], [Fellner&Latos&Tang'20]
- … homogeneous steady states, for cross–diffusion
  e.g. [Jüngel&Zamponi'17], [Daus&Jüngel&Tang'19]
- … inhomogeneous steady states, for diagonal linear diffusion
  e.g. [Di Francesco&Fellner&Markowich'08],
  [Hittmeir&Haskovec&Markowich&Mielke'18]

Our contribution

Seek non-negative unit-mass solutions $\boldsymbol{\rho} = (\rho_1, \rho_2)$ on $\mathbb{R}^d$ to

$$\partial_t \rho_1 = \Delta(\rho_1^m) + \nabla \cdot \big(\rho_1 \nabla [V_1 + \varepsilon \partial_1 H(\rho_1, \rho_2)]\big)$$
$$\partial_t \rho_2 = \Delta(\rho_2^m) + \nabla \cdot \big(\rho_2 \nabla [V_2 + \varepsilon \partial_2 H(\rho_1, \rho_2)]\big)$$

A Lyapunov functional is given by

$$\mathbf{E}^\varepsilon(\boldsymbol{\rho}) = \int_{\mathbb{R}^d} \left[\frac{\rho_1^m + \rho_2^m}{m-1} + V_1 \rho_1 + V_2 \rho_2 + \varepsilon H(\rho_1, \rho_2) \right] \mathrm{d}x - E^\varepsilon.$$

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where

- $m \geq 2$,
- H has "relatively bounded" derivates and is "relatively flat",
- $\nabla^2 V_j \geq \lambda \mathbf{1}$ with $\lambda > 0$.

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Theorem ([Beck&M&Zizza'22])

For all $\varepsilon \geq 0$ small enough:

- *unique stationary solution $\bar{\boldsymbol{\rho}} = (\bar{\rho}_1, \bar{\rho}_2)$;*
- *weak transient solution $\boldsymbol{\rho}$ for reasonable initial data $\boldsymbol{\rho}_0$;*
- $\mathbf{E}^\varepsilon(\boldsymbol{\rho}(t)) \leq \mathbf{E}^\varepsilon(\boldsymbol{\rho}(0)) \exp\big(-2\lambda(1 - K\varepsilon)t\big)$.

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e.g. where

- $m \geq 2$,
- $H(\rho_1, \rho_2) = \left(\dfrac{\rho_1 \rho_2}{1 + \rho_1 + \rho_2}\right)^m$,
- $V_j(x) = \frac{\lambda}{2}|x - \bar{x}_j|^2$,

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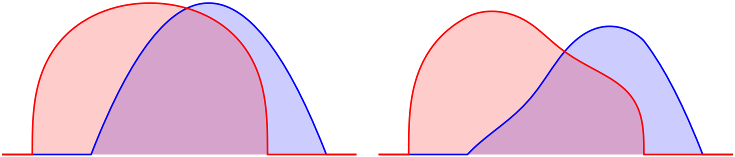
Steady states



Left: Steady state for $\varepsilon = 0$. Right: steady state for $\varepsilon > 0$.

Convexity (and total loss thereof)

The convexity behind it all

Recall: gradient flow in the L^2-Wasserstein metric W_2

$$\mathbf{L}(\rho) = \int_{\mathbb{R}^d} \left[\frac{\rho^m}{m-1} + V\rho\right] \mathrm{d}x - L \quad \rightsquigarrow \quad \partial_t \rho = \Delta(\rho^m) + \nabla \cdot (\rho \nabla V).$$

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Dissipation:

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{L}(\rho) = |\partial \mathbf{L}|^2(\rho) = \int_{\mathbb{R}^d} \rho \left| \nabla \left[\tfrac{m}{m-1} \rho^{m-1} + V \right] \right|^2 \mathrm{d}x.$$

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Theorem ([Otto'01,Carrillo&Toscani'00,Dolbeault&DelPino'02])

If $\nabla^2 V \geq \lambda$, then $\mathbf{L}(\rho(t)) \to 0$ at exponential rate 2λ because

$$|\partial \mathbf{L}|^2 \geq 2\lambda \mathbf{L}. \qquad (*)$$

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Theorem ([Otto'01,Carrillo&McCann&Villani'03,AGS'05])

$\nabla^2 V \geq \lambda \quad \Rightarrow \quad \mathbf{L}$ *is* *λ-uniformly displacement convex* $\quad \Rightarrow \quad (*)$

Gradient flow interpretation

$$\partial_t \rho_1 = \Delta(\rho_1^m) + \nabla \cdot \big(\rho_1 \nabla [V_1 + \varepsilon \partial_1 H(\rho_1, \rho_2)]\big)$$
$$\partial_t \rho_2 = \Delta(\rho_2^m) + \nabla \cdot \big(\rho_2 \nabla [V_2 + \varepsilon \partial_2 H(\rho_1, \rho_2)]\big)$$

is a gradient flow in the compound metric

$$\mathrm{d}(\boldsymbol{\rho}, \boldsymbol{\rho}')^2 = \mathrm{W}_2(\rho_1, \rho_1')^2 + \mathrm{W}_2(\rho_2, \rho_2')^2$$

for the energy

$$\mathbf{E}^\varepsilon(\boldsymbol{\rho}) = \int_{\mathbb{R}^d} \Big[\frac{\rho_1^m + \rho_2^m}{m-1} + V_1 \rho_1 + V_2 \rho_2 + \varepsilon H(\rho_1, \rho_2)\Big] \,\mathrm{d}x - E^\varepsilon.$$

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Question

$$|\partial \mathbf{E}^\varepsilon|^2 \geq 2\lambda^\varepsilon \mathbf{E}^\varepsilon \ ?$$

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Partial (yet quite discouraging) answer

Our assumptions make $\mathbf{E}^\varepsilon$ (flat) convex for each small $\varepsilon > 0$.
But: $\mathbf{E}_\varepsilon$ is not displacement semi-convex, not for any $\varepsilon > 0$.

Singular perturbation

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Lemma

For $\varepsilon = 0$: λ-uniform geodesic convexity.
For any $\varepsilon > 0$: failure of μ-uniform displacement convexity, for all $\mu \in \mathbb{R}$.

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Reason for loss of convexity: consider d-geodesics

$$[\tau_s \boldsymbol{\rho}](x) = \big(\rho_1(x - s\mathbf{e}), \rho_2(x)\big).$$

Choosing locally oscillatory ρ_1, ρ_2, the integral

$$\frac{\mathrm{d}^2}{\mathrm{d}^2 s}\bigg|_{s=0} \mathbf{E}^{\varepsilon}(\tau_s \boldsymbol{\rho}) \approx -\varepsilon \int_{\mathbb{R}^d} \partial_1 \partial_2 H(\boldsymbol{\rho})\, \partial_{\mathbf{e}} \rho_1\, \partial_{\mathbf{e}} \rho_2 \, \mathrm{d}x$$

can be made arbitrarily negative.

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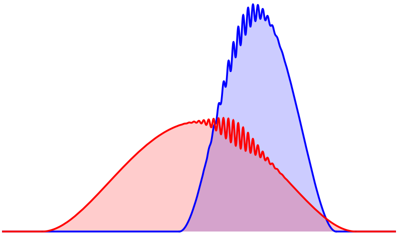

Put high frequency oscillations near $\hat{x}$ with $\partial_1\partial_2 H(\rho_1(\hat{x}), \rho_2(\hat{x})) \neq 0$.

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Corollary

No standard route to

$$|\partial \mathbf{E}^\varepsilon|^2 \geq 2\lambda \mathbf{E}^\varepsilon.$$

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For any $\varepsilon > 0$: failure of μ-uniform displacement convexity, for all $\mu \in \mathbb{R}$.

What about a change of metric?

- For certain reaction diffusion systems, long-time-asymptotics do follow via geodesic convexity, see e.g. [Liero&Mielke'13, Mielke&Mittnenzweig'18].
- Apparently, drift-(cross-)diffusion systems cannot be written as uniformly contractive gradient flows, see e.g. [Zinsl&M'15].

Realistic goal

$$|\partial \mathbf{E}^\varepsilon|^2 \geq 2\lambda \qquad\qquad \mathbf{E}^\varepsilon.$$

Realistic goal

$$|\partial \mathbf{E}^{\varepsilon}|^2 \geq 2\lambda(1 - K\varepsilon)\mathbf{E}^{\varepsilon}.$$

This seems no (trivial consequence of a) textbook inequality:

1. multiple components with nonlinear interaction,
2. ε-dependent, compactly supported minimizers,
3. on the whole $\mathbb{R}^d$.

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In a friendlier environment (bounded domain and $m = 1$),
such inequalities follow by "hands on" methods,
see e.g. [Alasio&Ranetbauer&Schmidtchen&Wolfram'20].

Coming up next

The technical slides

Transient solutions

Theorem

The time-discrete approximation via JKO,

$$\boldsymbol{\rho}^n := \operatorname*{argmin}_{\boldsymbol{\rho}} \left(\frac{1}{2\tau} \mathrm{d}(\boldsymbol{\rho}, \boldsymbol{\rho}^{n-1})^2 + \mathbf{E}^{\varepsilon}(\boldsymbol{\rho}) \right),$$

converge to a weak solution $\boldsymbol{\rho}^* = (\rho_1^*, \rho_2^*)$.

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Proof:

- Limit $\boldsymbol{\rho}^*$ exists by energy arguments. Really a weak solution?
- Solution concept requires derivatives of nonlinearities, weak convergence of $\nabla \partial_j H(\boldsymbol{\rho})$ needed.
- For a priori estimates, combine variations "along evolution" and "along heat flow",

$$\int_0^T \int_{\mathbb{R}^d} \rho_j |\nabla \rho_j^{m-1}|^2 \leq C \quad \text{and} \quad \int_0^T \int_{\mathbb{R}^d} e^{-\rho_j} |\nabla \rho_j^{m-1}|^2 \leq C.$$

Stationary solutions

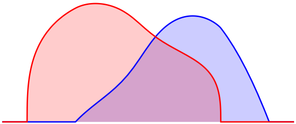
Theorem

There is a unique minimizer $\bar{\boldsymbol{\rho}} = (\bar{\rho}_1, \bar{\rho}_2)$ *of* $\mathbf{E}_\varepsilon$.
The support of $\bar{\rho}_j$ *and the* C^2*-norm of* $\partial_j H(\bar{\boldsymbol{\rho}})$ *are controlled, uniformly for small* $\varepsilon > 0$.

Proof:

- Existence by direct methods.
- $\bar{\rho}_j$'s support is $\{V_j \leq \bar{V}_j\}$.
- Euler-Lagrange equations are:

$$\bar{\rho}_1^{m-1} + \varepsilon \partial_1 H(\bar{\boldsymbol{\rho}}) = (\bar{V}_1 - V_1)_+$$
$$\bar{\rho}_2^{m-1} + \varepsilon \partial_2 H(\bar{\boldsymbol{\rho}}) = (\bar{V}_2 - V_2)_+$$



Regularity via IFT.

Instead of a linearization

Introduce the "convex expansion" of

$$\mathbf{E}^\varepsilon(\boldsymbol{\rho}) = \int_{\mathbb{R}^d} \Big[\frac{\rho_1^m + \rho_2^m}{m-1} + V_1\rho_1 + V_2\rho_2 + \varepsilon H(\rho_1,\rho_2)\Big]\,\mathrm{d}x$$

around the steady state $\bar{\rho}$:

$$\mathbf{L}^\varepsilon(\boldsymbol{\rho}) := \int_{\mathbb{R}^d} \Big[\frac{\rho_1^m}{m-1} + V_1^\varepsilon\rho_1 + \frac{\rho_2^m}{m-1} + V_2^\varepsilon\rho_2\Big]\,\mathrm{d}x - L^\varepsilon$$

with $V_j^\varepsilon := V_j + \varepsilon\partial_j H(\bar{\boldsymbol{\rho}})$.

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with $V_j^{\varepsilon} := V_j + \varepsilon\partial_j H(\bar{\boldsymbol{\rho}})$.

Corollary

$\nabla^2 V_j^{\varepsilon} \geq 2\lambda(1 - K_1\varepsilon)\mathbf{1}$, *and so* $|\partial\mathbf{L}^{\varepsilon}|^2 \geq 2\lambda(1 - K_1\varepsilon)\mathbf{L}^{\varepsilon}$.

Splitting

Split

$$\mathbf{E}^{\varepsilon}(\boldsymbol{\rho}) = \int_{\mathbb{R}^d} \left[\frac{\rho_1^m + \rho_2^m}{m-1} + V_1\rho_1 + V_2\rho_2 + \varepsilon H(\boldsymbol{\rho}) \right] \mathrm{d}x - E^{\varepsilon}$$
$$= \mathbf{L}^{\varepsilon}(\boldsymbol{\rho}) + \varepsilon \mathbf{A}^{\varepsilon}(\boldsymbol{\rho})$$

with

$$\mathbf{A}^{\varepsilon}(\boldsymbol{\rho}) = \int_{\mathbb{R}^d} \left[H(\boldsymbol{\rho}) - H(\bar{\boldsymbol{\rho}}) - (\boldsymbol{\rho} - \bar{\boldsymbol{\rho}}) \cdot \mathrm{D}H(\bar{\boldsymbol{\rho}}) \right] \mathrm{d}x.$$

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with

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Then:

$$\begin{aligned}|\partial \mathbf{E}^\varepsilon|^2 &\geq (1 - K_2\varepsilon)|\partial \mathbf{L}^\varepsilon|^2 - \frac{\varepsilon}{K_2}|\partial \mathbf{A}^\varepsilon|^2 \\ &= (1 - 2K_2\varepsilon)|\partial \mathbf{L}^\varepsilon|^2 + \frac{\varepsilon}{K_2}\left(K_2^2|\partial \mathbf{L}^\varepsilon|^2 - |\partial \mathbf{A}^\varepsilon|^2\right) \\ &\geq (1 - 2K_2\varepsilon) \cdot 2\lambda(1 - K_1\varepsilon)\mathbf{L}^\varepsilon + 0 \\ &\geq 2\lambda(1 - K\varepsilon)\mathbf{E}^\varepsilon.\end{aligned}$$

A similar story

A chemotaxis model

$$\partial_t \rho = \Delta \rho^2 + \nabla \cdot \left(\rho \nabla \left[V + \varepsilon \phi(c) \right] \right)$$
$$\partial_t c = \Delta c - \kappa c - \varepsilon \phi'(c)\, \rho.$$

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is the gradient flow of

$$\mathbf{E}^\varepsilon(\rho, c) = \int_{\mathbb{R}^d} \left[2\rho^2 + V\rho + \frac{1}{2}|\nabla c|^2 + \frac{\kappa}{2} c^2 + \varepsilon \phi(c) \rho\right] \mathrm{d}x$$

w.r.t. the compound metric

$$\mathrm{d}\big((\rho, c), (\rho', c')\big)^2 = \mathrm{W}_2(\rho, \rho')^2 + \|c - c'\|_{L^2}^2.$$

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is the gradient flow of

$$\mathbf{E}^\varepsilon(\rho,c) = \int_{\mathbb{R}^d}\left[2\rho^2 + V\rho + \frac{1}{2}|\nabla c|^2 + \frac{\kappa}{2}c^2 + \varepsilon\phi(c)\rho\right]\mathrm{d}x$$

w.r.t. the compound metric

$$\mathrm{d}\big((\rho,c),(\rho',c')\big)^2 = \mathrm{W}_2(\rho,\rho')^2 + \|c-c'\|_{L^2}^2.$$

Theorem ([Zinsl&M'15])

Suppose $\nabla^2 V \geq \kappa\mathbf{1}$. Then for each $\varepsilon > 0$ small enough:

$$\mathbf{E}^\varepsilon(\rho(t),c(t)) \leq \mathbf{E}^\varepsilon\big(\rho(0),c(0)\big)\,\exp\big(2(\kappa-K\varepsilon)t\big).$$

It's over — for now!

The road ahead:

- Understand gap between failure of $|\partial \mathbf{E}_\varepsilon|^2 \geq 2\lambda(1 - K\varepsilon)\mathbf{E}_\varepsilon$ and failure of (flat) convexity.
- Repeat for non-local interaction $\iint K(x,y)\rho_1(x)\rho_2(y)\,\mathrm{d}x\,\mathrm{d}y$ in place of $\int [V_1\rho_1 + V_2\rho_2]\,\mathrm{d}x$.
- Pass from second to fourth order.

Thank you!