On a pathwise stochastic control problem

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Consider the action functional

$$S_t[Z] = E\int_t^T [\frac{1}{2}|DZ_s|^2 + V(Z_s)]ds + S(Z_T)$$

defined for diffusion processes of the form

$$dZ_s = \sqrt{\nu}dW_s + DZ_s ds, \quad Z_t = x$$

where $\nu > 0$, $DZ_s \equiv u(s, Z_s)$, u with some regularity, i.e the process is Markov.

If we minimise S the value function satisfies the HJB equation

$$\frac{\partial S}{\partial t} - \frac{1}{2}|\nabla S|^2 + \frac{\nu}{2}\Delta S + V = 0, \quad S(T) = S$$

and the critical drift $u = -\nabla S$ satisfies a Burgers' eq.

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \frac{\nu}{2}\Delta u - \nabla V = 0, \quad u(T) = -\nabla S$$

Well known relation with Optimal transport via Girsanov's theorem:

since the law of Z (Q) is abs. continuous w.r.t the law of the Brownian motion (with coeff. $\sqrt{\nu}$) R with Radon-Nikodym derivative

$$\exp\left(\sqrt{\nu}\int_t^T u(Z_s)dW_s - \frac{\nu}{2}\int_t^T |u(Z_s)|^2 ds\right)$$

in the optimal control problem above we are looking at the relative entropy

$$H(Q|R) = E^Q\left(\log\frac{dQ}{dR}\right) \simeq S$$

Here we consider a pathwise version of this problem.

Define the random action functional

$$S_{t,x}(Z,u) = \int_t^T (L(u(s,Z_s)) + V(Z_s))\,\mathrm{d}s + S(Z_T),$$
$$dZ_s = \sqrt{\nu}dW_s + u(s,Z_s)\mathrm{d}s, \quad Z_t = x \text{ and } 0 \le t \le s \le T$$

(u random) and the value process

$$U_t(x) = \text{ess inf}_{u \in \mathcal{U}} S_{t,x}(Z,u).$$

To guarantee that there is a strong solution to Z for each drift u, we define $\mathcal{U}$ as the set of all processes (not necessarily adapted) which are uniformly bounded and s.t $|u(t,x,\omega) - u(t',x',\omega')| \le C(|t-t'| + |x-x'| + |\omega - \omega'|)$ for some $C > 0$.

For every stochastic process h with time derivative in L^2 and such that $h(t) = 0$ (here $L(u) = \frac{1}{2}|u|^2$)

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} S(Z+\varepsilon h) = \int_t^T (u_s(Z_s).\dot{h}(s))ds + \int_t^T \nabla V(Z_s).h(s)dt + \nabla S(Z_T).h(T$$

$$d\Big(u_s(Z_s).h(s)\Big) = d(u_s(Z_s)\Big).h(s) + u_s(Z_s).\dot{h}(s)ds,$$

and since $h(t) = 0$ we have, for all h,

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} S(Z+\varepsilon h) = \int_t^T h(t).\left[-du_s(Z_s) + \nabla V(Z_s)\right] ds$$
$$+ \left[\nabla S(Z_T) - u_T(Z_T)\right] h(T)$$

So

$$d(u_s(Z_s)) = \nabla V(Z_s)ds, \quad u_T(Z_T) = \nabla S(Z_T)$$

We cannot expect u to be adapted in general.

Example: $L(u) = \frac{1}{2}|u|^2$ and $V \equiv 0$.

Then $u(s, Z_s) = \nabla S(Z_T)$ for $t \leq s \leq T$, and

$$Z(s) = x + \int_t^s \nabla S(Z_T)\mathrm{d}\tau + \int_t^s \sqrt{\nu}\mathrm{d}W(\tau).$$

So Z is not adapted to an increasing filtration up to time s. We shall use non-adapted stochastic calculus.

Consider:

Prob. space: $\Omega = \{\omega \in C([t,T];\mathbb{R}^n), \omega(t) = x, \omega$ continuous$\}$, P Wiener measure (law of Brownian motion)

Cameron-Martin (tangent) space:
$H = \{h : [t,T] \to \mathbb{R}^n : h(t) = 0, h$ is a.c. and $\int_t^T |\frac{d}{ds}h(s)|^2 ds < +\infty\}$.

If F is a random variable in Ω, define

$$D_h F(\omega) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[F(\omega + \epsilon h) - F(w)]$$

Since H is a Hilbert space, the derivative gives rise to a gradient operator, $\langle \nabla F, h \rangle_H = D_h F$.
Defining

$$D_s F(\omega) = \frac{d}{ds} \nabla F(\omega),$$

we have $D_h F = \int_t^T D_s F \frac{d}{ds} h(s) ds$.

The Itô (-Skorohod) integral of non necessarily adapted processes, $\int u\, dW$, is the L^p limit, when it exists, of sums

$$\sum_k \mathcal{M}_k(u)(W(s_{k+1}) - W(s_k)) - \frac{1}{s_{k+1} - s_k} \int_{s_k}^{s_{k+1}} \int_{s_k}^{s_{k+1}} D_s u_\tau \, ds d\tau$$

where

$$\mathcal{M}_k(u) = \frac{1}{s_{k+1} - s_k} \int_{s_k}^{s_{k+1}} u_\tau \, d\tau,$$

The Stratonovich (-Skorohod) integral $\int u \circ dW$ is the limit of sums

$$\sum_k \mathcal{M}_k(u)(W(s_{k+1}) - W(s_k))$$

Relation between the two integrals:

$$\int_t^T u_s \circ dW_s = \int_t^T u_s dW_s + \frac{1}{2}\int_t^T (\mathbb{D}u)_s ds,$$

$(\mathbb{D}u)_s = D_s^+ u_s + D_s^- u_s$, with

$$D_s^+ u_s = \lim_{\tau \to s^+} D_s u_\tau,\ \ D_s^- u_s = \lim_{\tau \to s^-} D_s u_\tau$$

As in the adapted case, Stratonovich integration obeys the rules of ordinary differential calculus.

Itô-Wentzell formula in the non-adapted case (Stratonovich version) by Ocone and Pardoux.

If $Z_s = Z_t + \int_t^s B_\tau \circ dW_\tau + \int_t^s A_\tau d\tau$ and
$F_s(x) = F_t(x) + \int_t^s H_\tau(x) \circ dW_\tau + \int_t^s G_\tau(x) d\tau$,

$$F_s(Z_s) = F_t(Z_t) + \sum_k \int_t^s \nabla F_\tau(Z_\tau) \cdot B_\tau^k \circ dW_\tau^k + \int_t^s \nabla F_\tau(Z_\tau) \cdot A_\tau d\tau + \int_t^s H_\tau(Z_\tau) \circ dW_\tau + \int_t^s G_\tau(Z_\tau) d\tau$$

Theorem. The value process U is continuous in $\mathbb{R}^n \times [t, T]$, Lipchitz in space uniformly in t and α-Hölder in t uniformly in x for $\alpha < \frac{1}{2}$. It is a viscosity solution of the terminal value problem (stochastic HJB equation with transport noise)

$$\begin{cases} dv(s,x) = -\sqrt{\nu}\nabla v(s,x) \circ dW(s) \\ \qquad\qquad + \Big(V(x) + \text{ess inf}_u \left(L(u) + u \cdot \nabla v \right) \Big) \mathrm{d}s, \\ v(T,x) = S(x) \end{cases}$$

or, in Itô(-Skorohod) form,

$$\begin{cases} dv(s,x) = -\sqrt{\nu}\nabla v(s,x).dW(s) \\ \qquad\qquad + \Big(V(x) + \text{ess inf}_u \Big(L(u) + u \cdot \nabla v \Big) + \frac{\sqrt{\nu}}{2} \mathbb{D}_s(\nabla v)(s,x) \Big) \mathrm{d}s, \\ v(T,x) = S(x). \end{cases}$$

Moreover the solution is unique.

Viscosity solutions: (sub + super)

Let Φ be the solution of $d\Phi = -\sqrt{\nu}\nabla\Phi \circ dW_s, \quad \Phi(T, x) = S(x)$,
$F(x, v) = V(x) + \text{ess inf}_u \,(L(u) + u \cdot \nabla v)$.

An upper (resp. lower) semi-continuous function v defined on $[0, T] \times \mathbb{R}^n$ is a viscosity sub-solution (resp. super-solution) if it is bounded from above (resp. from below) with terminal data satisfying $v(\cdot, T) \leq S(x)$ (resp. $v(\cdot, T) \geq S(x)$), and, whenever $\phi \in C_b^2(\mathbb{R}^n)$, $\delta = \delta(\phi) > 0$, $g \in C^1([0, T]), \Phi(s, x) \in C_b^2(\mathbb{R}^n)$, for $s \in (s_0 - \delta, s_0 + \delta)$, and the map $(s, x) \mapsto v(s, x) - \Phi(s, x) - g(s)$ attains a local maximum (resp. local minimum) at $(s_0, x_0) \in \mathbb{R}^n \times (s_0 - \delta, s_0 + \delta)$, then

$$-g'(s_0) \leq F(\nabla\Phi(s_0, x_0), x_0) \quad (\text{resp.} - g'(s_0) \geq F(\nabla\Phi(s_0, x_0), x_0)).$$

It follows from Bellman's optimality principle:

$$U_t(x) = \text{ess inf}_u \left\{ \int_t^{t+\delta} (L(u) + V(Z_s))\, ds \right\} + U_{t+\delta}(Z_{t+\delta})$$

and uniqueness from a comparison principle:

If u and v are a viscosity solution sub-solution and a super-solution respectively, then

$$\sup_{x \in B_R(0)} (u(x,s) - v(x,s))_+ = \sup_{x \in B_R(0)} (u(x,0) - v(x,0))_+$$

for all $s \in [0, T]$ and $\forall R > 0$.

The case $L(u) = \frac{1}{2}|u|^2$. The equation for the drift,

$$\begin{cases} du(s,x) = -\sqrt{\nu}\nabla u(s,x) \circ dW(s) \\ \qquad\qquad -[(u \cdot \nabla u)(s,x) - \nabla V(x)]\mathrm{d}s, \\ u(T,x) = \nabla S(x) \end{cases}$$

We can reverse time, defining $\tilde{u}(s,x) = u(T-t,x)$ and

$$\begin{cases} d\tilde{u}(s,x) = -\sqrt{\nu}\nabla \tilde{u}(s,x) \circ dW(s) \\ \qquad\qquad +[(\tilde{u} \cdot \nabla \tilde{u})(s,x) - \nabla V(x)]\mathrm{d}s, \\ \tilde{u}(t,x) = \nabla S(x) \end{cases}$$

(a forward problem).

Symmetries of the action (again for $L(u) = \frac{1}{2}|u|^2$):

Consider a smooth (possibly random) vector field $Y :]t, T[\times \mathbb{R}^n \to [t, T] \times \mathbb{R}^n$ of the form $Y(s, x) = (T(s), X(s, x))$. Denote by $\Phi_\epsilon = (\varphi^0_\epsilon, \varphi_\epsilon)$ the flow generated by Y:

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \varphi^0_\epsilon(t) = T(t), \quad \frac{d}{d\epsilon}\Big|_{\epsilon=0} \varphi_\epsilon(t, x) = X(t, x).$$

A vector field Y as above is a Lagrangian infinitesimal variation symmetry of the action functional S if its flow is conserved, i.e, $\forall t_1, t_2 \in [t, T], t_1 < t_2$ and every $\epsilon > 0$, we have, a.s.,

$$\int_{t_1}^{t_2} \Big(\frac{1}{2}|D(Z_s)|^2 + V(Z_s)\Big)\mathrm{d}s = \int_{\varphi^0_\varepsilon(t_1)}^{\varphi^0_\varepsilon(t_2)} \Big(\frac{1}{2}|D(\varphi_\varepsilon(Z_{(\varphi^0_\varepsilon)^{-1}(s)}))|^2 + V(\varphi_\varepsilon(Z_{(\varphi^0_\varepsilon)^{-1}(s)}))\mathrm{d}s.$$

On the critical drift we have

$$D\left(\langle X,u\rangle - T\left(\frac{1}{2}|u|^2 - V\right)\right)(s,Z_s) = \frac{\nu}{2}T(s)\Delta V(Z_s).$$

N. Bhauryal, A.B.C. and C. Oliveira, *Pathwise stochastic control and a class of stochastic partial differential equations*, https://arxiv.org/pdf/2301.09214.pdf