A relaxed Beckmann's problem for SPD valued measures and application to Full Waveform Inversion

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Full Waveform Inversion (FWI)

Objective: reconstruct the velocity model $v : D \subset \mathbb{R}^d \to \mathbb{R}_+$ of the acoustic wave equation

$$\frac{1}{v}\partial_t^2 p - \Delta p = s\,, \quad \text{in } [0, T] \times D$$

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The whole temporal evolution is used for the calibration $\to$ **Full Waveform Inversion**

Highly non-convex problem (cycle skipping)

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For $\mu, \nu \in \mathcal{M}(\Omega)$, $\lambda \in \mathbb{R}_+$:

$$KR(\mu, \nu) = \sup_{\phi} \left\{ \int_{\Omega} \phi(\mu - \nu)\,,\ |\nabla \phi| \leq 1, |\phi| \leq \lambda \right\}$$

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"Relaxed" transport: if $\mu, \nu \in \mathcal{P}(\Omega)$ and $\lambda = +\infty \implies KR(\mu, \nu) = \mathcal{W}_1(\mu, \nu)$

$\mathcal{D} = KR(\mu, \nu)$ where $\mu = d_{\text{pred}}[v], \nu = d_{\text{obs}} \longrightarrow$ good results[1]

[1] Métivier et al., 2016

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If $\int_{\Omega} \mu = \int_{\Omega} \nu$ and $\lambda \gg 1$ then

$$KR(\mu, \nu) = KR((\mu - \nu)_+, (\mu - \nu)_-) = \mathcal{W}_1((\mu - \nu)_+, (\mu - \nu)_-)$$

$\implies$ **not convex** with respect to shifts

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Relaxed Beckmann's problem for SPD valued measures

Multi-component data $u(\cdot, x_r)$ and $L : \mathbb{R}^n \to \mathbb{S}^n_+$

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For $\mu, \nu \in \mathcal{M}(\Omega; V_+)$, $\lambda \in \mathbb{R}_+$:

$$\inf_{\substack{\sigma \in \mathcal{M}(\Omega; V^n) \\ \delta \in \mathcal{M}(\Omega; V)}} \left\{ \int_\Omega |\sigma|_{V^d} + \lambda \int_\Omega |\delta|_V \,,\ \mathrm{div}_V(\sigma) = \mu - \nu + \delta \right\}$$

$V = \mathbb{S}^n, V_+ = \mathbb{S}^n_+, V^d = [\mathbb{S}^n]^d$,

$|\cdot|_V$ Frobenius norm, $|\cdot|_{V^d} = \sum_{k=1}^d |\cdot|_V$

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Choice of $p, q \in \{1, 2\}$:

- Sensitivity to (small) displacements (optimal transport problem) $\longrightarrow$ $p = 1$
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Choice of λ:

- For small values, $\mathcal{T}_{p,q}^{\lambda}$ behaves like an L^q distance of $\mu - \nu$
- For big values, $\mathcal{T}_{p,q}^{\lambda}$ behaves like an L^q distance on "the mass difference"
- Dimensional analysis:

$$[\mu] = [\nu] = A,\ [\sigma] = A \cdot l \quad \longrightarrow \quad [\lambda] = A^{p-q} \cdot l^p$$

where l is a length, A an amplitude

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$$\mathcal{T}^{\lambda}_{1,2}(\mu,\nu) = \sup_{\phi \in C(\Omega;V)} \left\{ \int_\Omega <\phi, \nu - \mu>_V - \frac{1}{2\lambda} \int_\Omega |\phi|^2_V \,,\ |\nabla\phi|_{V^n} \leq 1 \right\}$$

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$p = 2, q = 2$

$$\mathcal{T}^{\lambda}_{2,2}(\mu,\nu) = \sup_{\phi} -\frac{1}{2} \int_\Omega |\nabla \phi|^2_{V^n} - \frac{1}{2\lambda} \int_\Omega |\phi|^2_V + \int_\Omega \langle \phi, \nu - \mu \rangle_V$$

$\longrightarrow$ not a transport, not coupled...

Exact solutions for $p = 1, q = 1$

Consider two delta measures $\mu = M_1\delta_{x_1}, \nu = M_2\delta_{x_2},\ M_1, M_2 \in \mathbb{S}^n_+$

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The cost $\mathcal{T}^{\lambda}_{1,1}(\mu, \nu)$ is equal to ($\gamma = \frac{\lambda}{|x_1 - x_2|}$)

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\begin{cases}
\lambda(|M_1|_V + |M_2|_V) & \text{if } 1 + \dfrac{\langle M_1, M_2 \rangle_V}{|M_1|_V |M_2|_V} \leq \dfrac{1}{2\gamma^2} \\
\lambda |M_1 - M_2|_V + |x_1 - x_2||M_1|_V & \text{if } \dfrac{\langle M_1, M_1 - M_2 \rangle_V}{|M_1|_V |M_1 - M_2|_V} \leq -\dfrac{1}{2\gamma^2} \\
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|x_1 - x_2| \Big((\gamma^2 - \frac{1}{2}) |M_1 - M_2|^2_V + \frac{1}{2}(|M_1|^2_V + |M_2|^2_V) + & \\
\qquad (4\gamma^2 - 1)^{\frac{1}{2}} (|M_1|^2_V |M_2|^2_V - \langle M_1, M_2 \rangle^2_V)^{\frac{1}{2}} \Big)^{\frac{1}{2}} & \text{else.}
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$\gamma \leq \frac{1}{2} \implies$ the first condition is always met ($M_1, M_2 \geq 0$)

$\longrightarrow$ λ represents (half) the distance at which mass is transported

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$\longrightarrow$ λ represents the distance at which a unit of mass is transported

The tuning of λ can be done by choosing a reference transport distance and a reference mass to be transported

Numerical solution

Numerical solution of $\mathcal{T}_{p,q}^{\lambda}$:

- Finite difference discretization
- SDMM algorithm: primal-dual proximal splitting technique
- Bottleneck: solution of a Poisson equation
- The SPD transport does not reflect on the complexity of the algorithm
- $(p = 1, q = 2) \longrightarrow$ the higher regularity can be exploited (FISTA, Chambolle-Pock, ...)

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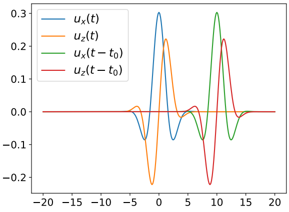
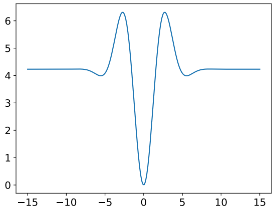
Solution of the inverse problem:

- l-BFGS algorithm: requires $\frac{\partial \mathcal{D}(\mu[v],\nu)}{\partial v}$
- Derivative via adjoint state method: requires $\frac{\partial \mathcal{T}_{p,q}^{\lambda}(\mu,\nu)}{\partial \mu}$
- thanks to the dual structure:

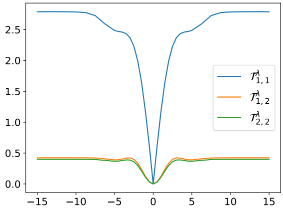
$$\frac{\partial \mathcal{T}_{p,q}^{\lambda}(\mu,\nu)}{\partial \mu} = \phi$$

Numerical results: sensitivity to time shift

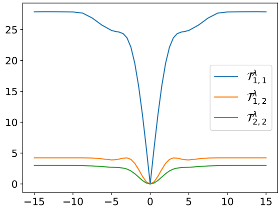




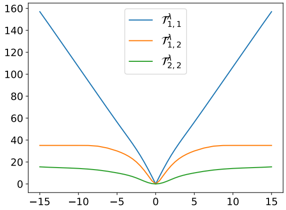
L^2 norm



$\lambda = 0.1$



$\lambda = 1$



$\lambda = 10$

Numerical results: Marmousi test case

$D = [a, b] \times [c, d] \subset \mathbb{R}^2, n = 2, T = 3000,\ N_r = 169$

$\Omega = \{x_1, .., x_{N_r}\} \times [0, T]$ semi-discrete 2d domain

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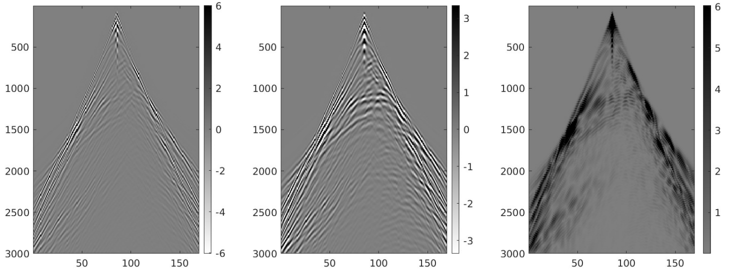

u_x $\quad$ u_z $\quad$ $\alpha = \sqrt{u_x^2 + u_z^2}$

"Dimensional anisotropy": $\tilde{\Omega} = \{x_r, .., x_{N_r}\} \times [0, T/\bar{v}]$ where $\bar{v}$ mean velocity

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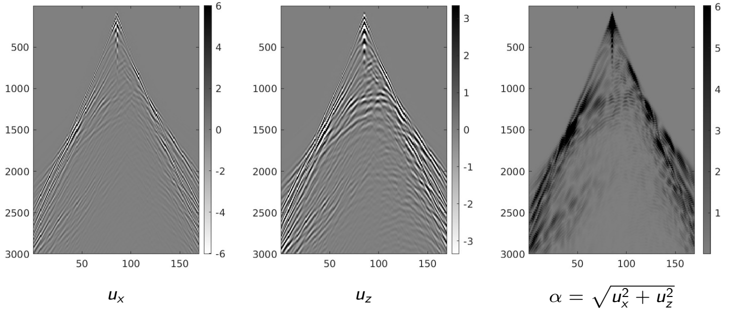
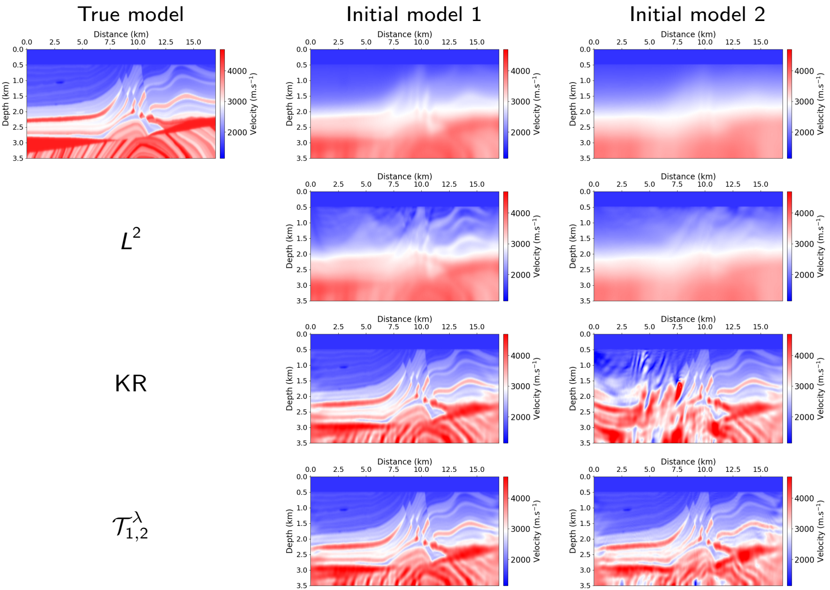


"Dimensional anisotropy": $\tilde{\Omega} = \{x_r, .., x_{N_r}\} \times [0, T/\bar{v}]$ where $\bar{v}$ mean velocity

For $(p = 1, q = 1)$, $\lambda \approx \frac{fq}{2\bar{v}}$; for $(p = 1, q = 2)$, $\lambda \approx \frac{fq}{2\bar{v}10^{-4}}$; $\;fq$: frequency of s

Numerical results: Marmousi test case



Conclusions

We introduced an (unbalanced) L1 transport for SPD valued measures to transport multi-component signals

Computationally affordable

Good results for the FWI problem justifying the approach

Applications to other problems can be foreseen

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BUT:

- Rely on a lift function: linked to the physics of the problem at hand
- Sensitivity to the calibration of λ (depending again on the lift/physics)

Thank you for your attention!