The mathematical description of large scale atmospheric flows

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Optimal Transport and Applications to Physics, 2023

The dynamics of large scale flows

Weather forecast: based on space-time averaged version of the Navier-Stokes equations, yet pretty accurate $\Longrightarrow$ the large scale dynamics must control the weather patterns to a large extent.

A specific reduced system of equations valid on large scales are the **semi-geostrophic equations**. These equations

- capture the main features of the dynamics
- are amenable to mathematical analysis and robust numerical computations
- admit solutions modelling singular behaviour (atmospheric fronts) so can be solved past the front formation

Semi-geostrophic equations

Assumptions:

- no viscosity (ok for atmosphere)
- Boussinesq approximation: negligible density variations unless multiplied by g
- Shallow atmosphere (the *variable* Coriolis force has no effect in the vertical direction)
- Hydrostatic balance: density is proportional to vertical pressure variation
- geostrophic balance (valid for strong Coriolis forcing): the horizontal pressure gradient balances the Coriolis force. The geostrophic velocity (with f=rotation coefficient) is

$$\mathbf{v}^g = (-\frac{1}{f}\partial_2 p, \frac{1}{f}\partial_1 p, 0).$$

The semigeostrophic equations are a second-order approximation to the Euler equations - they conserves energy and are valid for large scales - f can be variable.

Semi-geostrophic equations - 3D incompressible case

$$
\begin{aligned}
&(\partial_t + \mathbf{u}\cdot\nabla)(v_1^g, v_2^g) + (\partial_1 p, \partial_2 p) = (u_2, -u_1) \\
&(\partial_t + \mathbf{u}\cdot\nabla)\rho = 0, \quad \rho = -\partial_3 p, \\
&(v_1^g, v_2^g) = (-\partial_2 p, \partial_1 p), \qquad \mathbf{x} \in \Omega \subset \mathbb{R}^3 \\
&\nabla \cdot \mathbf{u} = 0.
\end{aligned}
$$

unknowns: $\mathbf{u} = (u_1, u_2, u_3)$; $\mathbf{v}^g = (v_g^1, v_g^2, 0)$; p; ρ.

Solutions conserve the **geostrophic energy**

$$
E(t) = \int_\Omega \left\{ \frac{1}{2}\left[(v_1^g)^2 + (v_2^g)^2\right] + \rho x_3 \right\} d\mathbf{x}
$$

Geostrophic formulation of the incompressible system

Achieved via a change of space variable $T : \Omega \to \mathcal{Y} \subset \mathbb{R}^3,\ T\mathbf{x} = \mathbf{y}$
- must be well defined and invertible.

Originally, Hoskins' geostrophic variable change:
$P(t,\mathbf{x}) = p(t,\mathbf{x}) + \frac{1}{2}(x_1^2 + x_2^2)$ *and* $\mathbf{x} \to \mathbf{y}(t,\mathbf{x}) = \nabla P(t,\mathbf{x})$

The equations become

$$\begin{aligned}
&\partial_t \nu + \nabla \cdot (\mathbf{U}\nu) = 0,\\
&\nu = T\#\chi_\Omega, \quad (T = \nabla P),\\
&\mathbf{U} = J(Id - T^{-1}), \quad J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

$\mathbf{U} = \mathbf{v}^g$ is the geostrophic velocity.

Energy minimisation and optimal transport

$$\partial_t \nu + \mathbf{U} \cdot \nabla \nu = 0, \quad \mathbf{U} = J(Id - T^{-1}), \quad \nu = T \# \chi_\Omega.$$

The nonlinear evolution for ν is not determined - need to determine $\mathbf{U}$ or, equivalently, have a selection principle for T.

Energy:

$$E_t(T) = \int_\Omega \left[\frac{1}{2}(x_1 - T_1)^2 + \frac{1}{2}(x_2 - T_2)^2 - x_3 T_3 \right] d\mathbf{x}$$

Energy minimisation $\sim$ T is the optimal transport map wrt the (quadratic) cost

$$c_2(x, y) = \left[\frac{1}{2}(x_1 - y_1)^2 + \frac{1}{2}(x_2 - y_2)^2 - x_3 y_3 \right].$$

Hence $T = \nabla P$, P convex

(*Cullen convexity principle* and *Brenier's polar factorisation*)

Geostrophic formulation of the compressible system

The compressible system has an analogous formulation in geostrophic variables: with $T : \Omega \to \mathcal{Y} \subset \mathbb{R}^2 \times (\varepsilon, \frac{1}{\varepsilon})$,

$$\partial_t \alpha + \nabla \cdot (\mathbf{U}\alpha) = 0;$$
$$\alpha = T\#\sigma, \quad T \textit{ optimal}$$
$$U = J(Id - T^{-1}) \ (\Longrightarrow \nabla \cdot U = 0);$$

The source measure $\sigma = \theta\rho$ is an unknown of the problem

$T =$ optimal transport map from σ to α with cost

$$c_{com}(x, y) = \frac{\left[\frac{1}{2}(x_1 - y_1)^2 + \frac{1}{2}(x_2 - y_2)^2 + \Phi(x)\right]}{y_3}.$$

($\Phi(x)$ is the given geopotential, here $= -x_3$)

Geostrophic energy and minimisation

Incompressible system: The minimum of the energy is

$$E_t(\nu) = \inf_{T : T\#\chi_\Omega = \nu} \int_\Omega \left[\frac{1}{2}(x_1 - T_1)^2 + \frac{1}{2}(x_2 - T_2)^2 - x_3 T_3 \right] dx,$$

Compressible system: The minimum of the energy is

$$E_t(\sigma; \alpha) = \inf_{T : T\#\sigma = \alpha} \int_\Omega \frac{\left[\frac{1}{2}(x_1 - T_1)^2 + \frac{1}{2}(x_2 - T_2)^2 + \Phi(x)\right]}{T_3} \,\mathrm{d}\sigma(\mathbf{x})$$
$$+ \kappa \int_\Omega \sigma^\gamma \,\mathrm{d}\mathbf{x}, \qquad \sigma = \arg\min_{\mathscr{P}_{\mathrm{ac}}(\Omega)} E(\sigma; \alpha),$$

($\gamma \in (1, 2)$ is the ratio of specific heats and κ is a constant)

Incompressible vs compressible

Incompressible:	Compressible:
$\partial_t \nu + \nabla \cdot (\mathbf{U}\nu) = 0$ $\nu = T\#\chi_\Omega$; T optimal for c_2 $\mathbf{U} = J(Id - T^{-1})$	$\partial_t \alpha + \nabla \cdot (\mathbf{U}\alpha) = 0$ $\alpha = T\#\sigma$, T optimal for c_{com} $\mathbf{U} = J(Id - T^{-1})$
Energy minimisation: $E = \inf_{T:T\#\chi_\Omega=\nu} \int_\Omega c_2(\mathbf{x}, T\mathbf{x})d\mathbf{x}$	Energy minimisation: $E = \inf_{T:T\#\sigma=\alpha} \int_\Omega c_{com}(\mathbf{x}, T\mathbf{x})\,\mathrm{d}\sigma(\mathbf{x})$ $+\kappa \int_\Omega \sigma^\gamma \,\mathrm{d}\mathbf{x}$ $\sigma = \arg\min_{\mathscr{P}_{\mathrm{ac}}(\Omega)} E(\sigma;\alpha)$
$E = E_t(\nu)$	$E = E_t(\sigma;\alpha)$

BB's existence proof strategy (incompressible)

$T_t = \nabla P_t$ - optimal transport map *at each fixed time t*
ν - must satisfy (weakly) the transport equation

$$\partial_t \nu + \nabla \cdot (\mathbf{U}\nu) = 0, \quad \mathbf{U}(t, \mathbf{y}) = J(\mathbf{y} - \nabla P^*(t, \mathbf{y}))$$

with velocity $\mathbf{U}(t, X) = J(\mathbf{y} - \nabla P^*(t, \mathbf{y}))$ *not* Lipschitz - only BV

Time stepping argument $(h = \Delta t)$:

Assume at $t_k = kh$, P_k convex and $\alpha_k = \nabla P_k \# \chi_\Omega$

- define $\mathbf{U}_k(\mathbf{y}) = J(\mathbf{y} - \nabla P_k^*)$
- solve $\partial_t \alpha + \nabla \cdot (\mathbf{U}_k \alpha) = 0$ for $t \in (kh, (k+1)h)$

(needs regularisation)

- define $\alpha_{k+1} = \alpha(\cdot, t(k+1))$
- set P_{k+1} = solution of optimal transport from χ_Ω to α_{k+1}

then take regularisation and $h \to 0$ limit.

Semi-discrete optimal transport approach - incompressible

Proof using space rather than time discretisation - *semi-discrete optimal transport* techniques:

1. discrete geostrophic solutions with *well-prepared* initial data given by a discrete probability measure exist, are unique, and are defined by trajectories that are twice continuously differentiable in time;
2. Lipschitz-in-time solutions of SG in geostrophic coordinates with arbitrary compactly-supported initial probability measure can be constructed as the uniform limit of a sequence of *discrete geostrophic solutions* as in 1.

Note: numerical geometric method for the Eady slice based on semi-discrete OT (Cotter *& al* 2018)

Well-prepared means: seeds lie in distinct horizontal planes

Space discretisation, aka the *geometric method*

$$\partial_t \nu + \mathrm{div}(\nu \mathbf{U}) = 0 \quad \text{in } \mathbb{R}^3 \times (0,\tau), \qquad \nu(t=0) = \nu_0$$

$$\begin{array}{ccc}
\nu_0 \in \mathscr{P}_{\mathrm{ac}}(\mathcal{Y}) & \overset{\text{PDE for } \nu}{\longrightarrow} & \nu(t) \\
\downarrow & & \uparrow \quad N \to \infty \\
\nu_0^N = \dfrac{1}{N}\displaystyle\sum_{i=1}^N \delta_{z_0^i} & \overset{\text{ODE for } z_t^i}{\longrightarrow} & \nu_t^N = \dfrac{1}{N}\displaystyle\sum_{i=1}^N \delta_{z_t^i}
\end{array}$$

well-prepared, $\nu_0^N \to_{N\to\infty} \nu_0$

$$\dot{z}_t^i = u_i$$

$(\mathbf{z}_t = (z_t^1, .., z_t^N) \in (R^3)^N$ seed vector defining $\nu_t^N)$

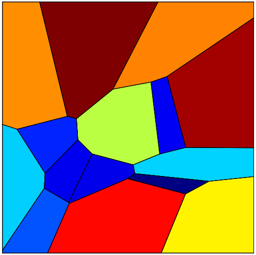
How to define the transport velocity $u_i \sim$ semi-discrete OT

For ν_t^N defined by the seed vector $\mathbf{z}$ of N points, weights $m_i (= \frac{1}{N})$

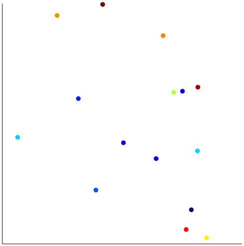
$$T^N = \text{optimal transport map from } \mathcal{L}_\Omega \text{ to } \nu_t^N \text{ with quadratic cost}$$

$$= \underset{T:\Omega\to\mathbb{R}^3}{\arg\min} \left\{ \int_\Omega |T^N(x) - x|^2 \, \mathrm{d}x \, : \, T^N \# \chi_\Omega = \nu_t^N \right\}$$

$$T^N \text{ must be of the form } T^N = \sum_{i=1}^N z^i C_i, \quad C_i = \text{ tessellation of } \Omega, \ |(C_i)| = m_i .$$



$\xrightarrow{T_t^N}$



Laguerre tessellations and the nonlocal transport velocity u_i

Optimal transport $=$ optimal partition problem solved by Laguerre cells $C_i(\mathbf{z},\mathbf{w})$,

$$C_i(\mathbf{z},\mathbf{w}) = \{\mathbf{x} \in \Omega : |x - z_i|^2 - w_i \leq |x - z_j|^2 - w_j,\ \forall j = 1, .., N\}.$$

with $\mathbf{w}$ optimal weight$=$ maximiser of the Kantorovich functional

$$\mathbf{w} = \max_{\mathbf{w}\in\mathbb{R}^N} \sum_{i=1}^{N} \left[m_i w_i + \int_{C_i(\mathbf{z},\mathbf{w})} [(\mathbf{x} - z^i)^2 - w_i] d\mathbf{x} \right].$$

Then the discrete transport velocity u_i is

$$u_i(t) := \mathbf{U}(y,t)\Big|_{y=z_t^i} = J(z_t^i - c_i(t))$$

where $c_i(t)$ is the centroid of $C_i(\mathbf{z},\mathbf{w}) = (T_t^N)^{-1}(\{z_t^i\})$:

$$c_i(t) = \frac{1}{|C_i(\mathbf{z},\mathbf{w})|} \int_{C_i(\mathbf{z},\mathbf{w})} \mathbf{x} d\mathbf{x}$$

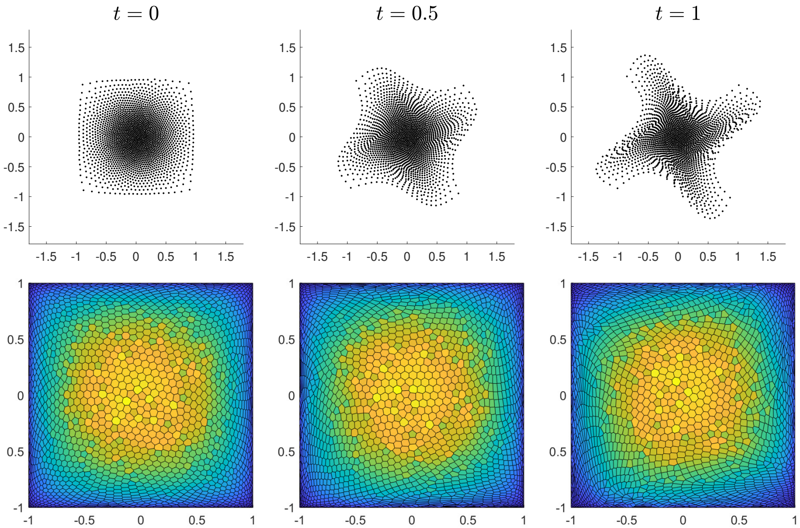
Statement of the existence results - incompressible

Theorem 1
Let $\Omega \subset \mathbb{R}^3$ be open, bounded and convex, and $N \geq 2$. For any $\tau > 0$ and any well-prepared discrete probability measure ν_0^N, there exists a unique $z^N \in C^2([0,\tau],(\mathbb{R}^3)^N)$ such that $\nu_t^N \in C([0,\tau],\mathscr{P}(\mathbb{R}^3))$ is a discrete geostrophic solution with initial measure ν_0^N. Moreover, this solution is energy-conserving.

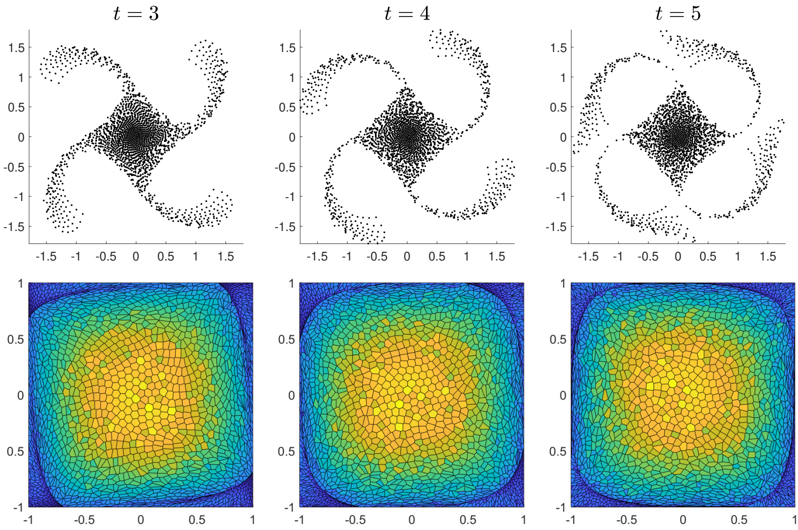
Theorem 2 (*cfr.* Loeper, Feldman-Tudorascu)
Let $\Omega \subset \mathbb{R}^3$ be open, bounded and convex,. For any $\tau > 0$ and any $\nu_0 \in \mathscr{P}(\mathbb{R}^3)$, there exists an energy-conserving geostrophic solution $\nu_t \in C^{0,1}([0,\tau],\mathscr{P}(\mathbb{R}^3))$ with initial measure ν_0, obtained as the limit in W_2 of a sequence ν_t^N of discrete solutions as in Thm 1:

$$\lim_{N\to\infty} \sup_{t\in[0,\tau]} W_2(\nu_t^N,\nu_t) = 0.$$

2D Gaussian initial condition - seeds and cells



2D Gaussian initial condition - seeds and cells



Discrete solutions of the compressible system - what changes?

The compressible system has associated geostrophic energy

$$E(\sigma;\alpha) = \inf_{T:T\#\sigma=\alpha} \int_\Omega c_{com}(\mathbf{x}, T\mathbf{x})\,\mathrm{d}\sigma(\mathbf{x}) + \kappa \int_\Omega \sigma^\gamma\,\mathrm{d}\mathbf{x},$$

with $$c_{com}(\mathbf{x}, T\mathbf{x}) = \frac{\left[\frac{1}{2}(x_1 - T_1)^2 + \frac{1}{2}(x_2 - T_2)^2 - x_3\right]}{T_3}$$

(*non-symmetric, twisted cost*) and is given by

$$\begin{aligned}
&\partial_t\alpha + \nabla\cdot(\mathbf{U}\alpha) = 0\\
&\alpha = T\#\sigma, \quad T \text{ optimal for } c_{com}\\
&\mathbf{U} = J(Id - T^{-1}),\\
&\sigma = \operatorname*{arg\,min}_{\mathscr{P}_{\mathrm{ac}}(\Omega)} E(\sigma;\alpha).
\end{aligned}$$

we can prove the same results, but with significant technical differences to overcome in the proof.

c-Laguerre cells and optimal transport

Given a seed vector $\mathbf{z} \in (\mathbb{R}^3)^N$ and a weight vector $\boldsymbol{w} \in \mathbb{R}^N$, the c-Laguerre cells L_i^c are

$$L_i^c(\mathbf{z}, \boldsymbol{w}(\mathbf{z})) = \{\mathbf{x} \in \Omega : c(\mathbf{x}, \mathbf{z}_i) - w_i(\mathbf{z}) \leq c(\mathbf{x}, \mathbf{z}_j) - w_j(\mathbf{z}) \quad \forall\, j = 1, \ldots, N\}.$$

The optimal weight $\mathbf{w}^*$ coupled with the minimiser σ of the energy defines the optimal semi-discrete transport

$$T^N = \sum_{i=1}^{N} z_i \chi_{L_i^c(\mathbf{z}, \mathbf{w}^*)}, \qquad \sigma(L_i^c) = m_i (= \frac{1}{N})$$

and the optimal centroid map $(\sim (T^N)^{-1})$

$$\mathbf{C}(\mathbf{z}, \mathbf{w}^*) = (c_1, \ldots c_N), \qquad c_i(\mathbf{z}, \mathbf{w}^*) = \frac{1}{\sigma(L_i^c(\mathbf{z}, \mathbf{w}^*(\mathbf{z})))} \int_{L_i^c(\mathbf{z}, \mathbf{w}^*(\mathbf{z}))} \mathbf{x} \,\mathrm{d}\sigma(\mathbf{x})$$

Optimal transport problem $\Longleftrightarrow$ optimal weight $\mathbf{w}^*$ and $\underset{\sigma \in \mathscr{P}_{\mathrm{ac}}(\Omega)}{\arg\min}\ E$

Optimal pair $(\mathbf{w}, \sigma)$, given $\alpha^N \sim \mathbf{z} \in (\mathbb{R}^3)^N$

Energy functional ($\mathcal{T}_c(\sigma, \alpha^N)$=optimal transport map for c_{com}):

$$E(\sigma, \alpha^N) = \mathcal{T}_c(\sigma, \alpha^N) + \int_\Omega f(\sigma(\mathbf{x}))\,\mathrm{d}\mathbf{x}\,, \quad f(s) = \kappa|s|^\gamma.$$

The dual energy functional can be shown to be (*cfr. Sarrazin*)

$$g(\boldsymbol{w}, \alpha^N) = \sum_{i=1}^N \left[m_i w_i - \int_{L_i^c(\mathbf{z},\boldsymbol{w})} f^*(w_i - c(\mathbf{x}, \mathbf{z}_i))\,\mathrm{d}\mathbf{x} \right].$$

Optimality condition for $(\mathbf{w}, \sigma)$:

$$\int_{L_i^c(\mathbf{z},\boldsymbol{w})} (f^*)'(w_i - c(\mathbf{x}, \mathbf{z}_i)\,\mathrm{d}\mathbf{x} = m_i, \qquad \sigma(\mathbf{x}) = (f^*)'(w_i - c(\mathbf{x}, \mathbf{z}_i)).$$

f^*= Legendre-Fenchel transform of f

Differentiability of the discrete transport velocity

$u_i(t) := J(z_t^i - c_i(t))$, where

$$c_i(\mathbf{z}, \mathbf{w}^*) = \frac{1}{\sigma\left(L_i^c(\mathbf{z}, \mathbf{w}^*(\mathbf{z}))\right)} \int_{L_i^c(\mathbf{z}, \mathbf{w}^*(\mathbf{z}))} \mathbf{x} \, \mathrm{d}\sigma\left(\mathbf{x}\right)$$

so differentiability properties depend on differentiability wrt **z** of the centroid map and of the weight - namely that these maps are C^1.

Crucial geometric properties of the c-Laguerre tessellation: the cost c_{com} satisfies the conditions of De Gournay *et al* (provided Ω does not contain any section of a paraboloid of the form $x_3 = -\frac{1}{2}(x_1^2 + x_2^2) + L(x_1, x_2)$)

Note: c_{com} does not satisfy the regularity conditions due to Loeper, which would imply those of De Gournay *et al*

c-Laguerre cells (vertical slices in the x_1, x_3 plane)

Figure: 5 c-Laguerre cells

Figure: 10 c-Laguerre cells

c-Laguerre cells - cont

Figure: 25 c-Laguerre cells

Figure: 50 c-Laguerre cells

Conclusions

Using these properties, we can prove the differentiability of centroid and optimal weight maps wrt $\mathbf{z}$, and then the proof of both results is then analogous to the one for the incompressible system.

We can also prove that the discrete solutions conserve energy

- Basis for effective numerical schemes (incompressible Eady slice, shallow water - 3D in progress) that are energy-conserving;
- Possible basis for an explicit and intuitive connection between geostrophic coordinates and corresponding flows in the physical domain.

Appendix: the geometric conditions of De Gournay *et al*

1. For all $1 \leq i \leq n$, $(\mathbf{x}, \mathbf{y}) \mapsto c(\mathbf{x}, \mathbf{y}^i)$ is $W^{2,\infty}(\mathcal{X} \times B(\mathbf{y}_0, r))$, where $B(\mathbf{y}_0, r)$ is a ball around the point $\mathbf{y}_0$.
2. There exists $\varepsilon > 0$ such that for all $1 \leq k \neq i \leq n$, $\forall\, \mathbf{x} \in e^{ik}$
$$\left\|\nabla_{\mathbf{x}} c(\mathbf{x}, \mathbf{y}^i) - \nabla_{\mathbf{x}} c(\mathbf{x}, \mathbf{y}^k)\right\| \geq \varepsilon.$$
3. For all i, there exists $s > 0$ and $C > 0$ such that for all $0 \leq k \neq j \leq n$ and $\varepsilon, \varepsilon' \in (0, s)$, it holds
$$\left|\mathcal{N}_{ik}(\varepsilon) \cap \mathcal{N}_{ij}(\varepsilon')\right| \leq C\varepsilon\varepsilon' \quad \lim_{\varepsilon \to 0} \mathcal{H}^{d-1}\left(e^{ik} \cap \mathcal{N}_{ij}(\varepsilon)\right) = 0,$$
for $\mathcal{N}_{ik}(\varepsilon)$ the ε thickening of the edge e^{ik}.
4. There exists $C > 0$ such that for all i, j
$$\mathcal{H}^{d-1}(e^{ij} \cap \mathcal{X}) \leq C.$$