Theory and Approximate Solvers for

Branched Optimal Transport

with Multiple Sources

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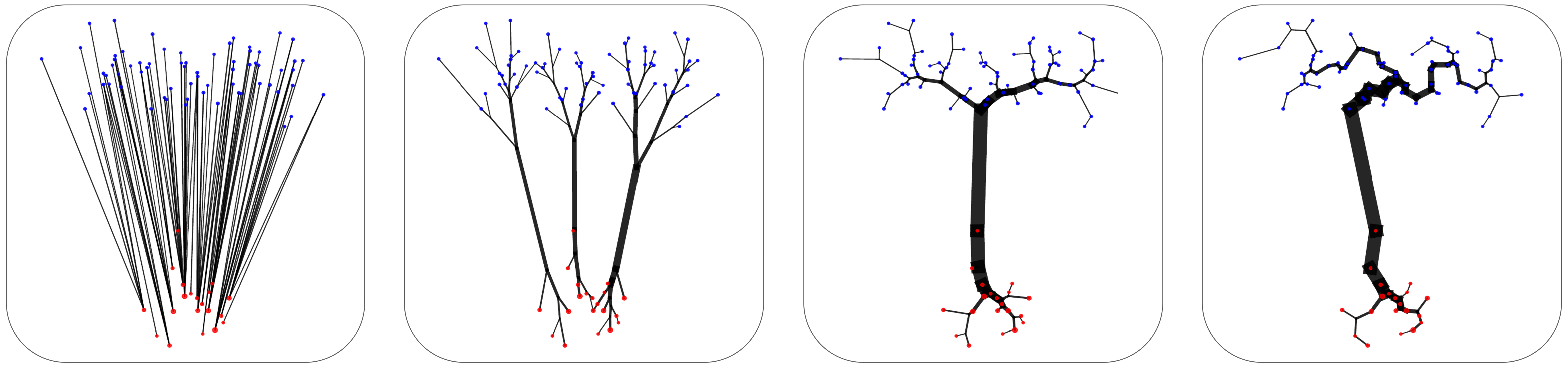
Optimal transport with branched transportation routes.



$\alpha = 1$
Optimal Transport

$\alpha = 0.95$

$\alpha = 0.5$

$\alpha = 0$
Euclidean Steiner Tree

Sinks with given demands

Branching points with flow conservation

Sources with given supplies

What makes it *branched*?

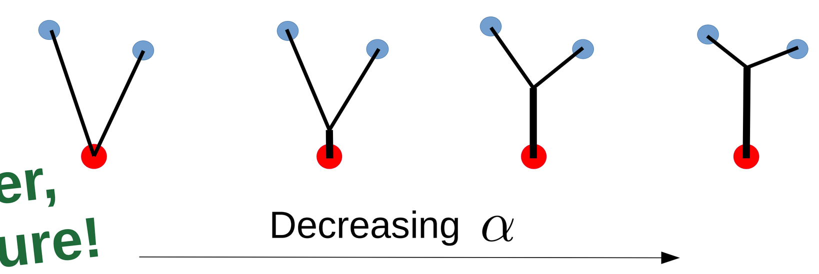
The BOT cost function is subadditive w.r.t. the transported mass [1]:

Transportation Cost

$$\mathcal{C} = \sum_{\substack{\text{all edges} \\ e=(i,j)}} m_{ij}^{\alpha} \|x_i - x_j\|_2$$

Subadditivity encourages branching:

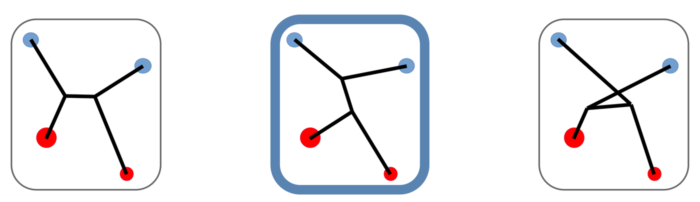
$$(m_1 + m_2)^{\alpha} < m_1^{\alpha} + m_2^{\alpha} \quad \text{for} \quad \alpha \in [0, 1)$$



One extra parameter, plenty more structure!

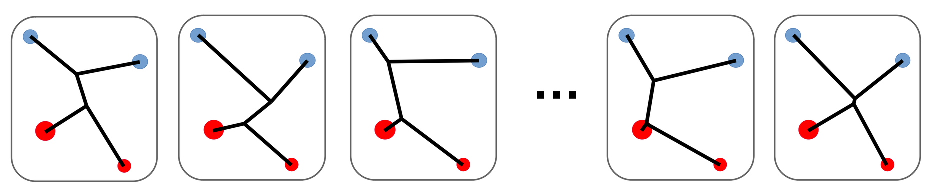
BOT optimization: 2 in 1

1) Discrete choice between different topologies:



- **Combinatorial**
- Super-exponentially many topologies

2) Given a topology, find the optimal branching point positions:



- **Continuous**
- Convex but non-differentiable

Highlights

- ➔ combinatorial and continuous optimization
- ➔ fascinating, non-trivial structure
- ➔ simple instructive model for branched transportation
- ➔ new machine learning playground

OUR CONTRIBUTIONS:

- ➔ new theoretical insights on structural properties
- ➔ new efficient BOT solver

The theory of BOT in a nutshell

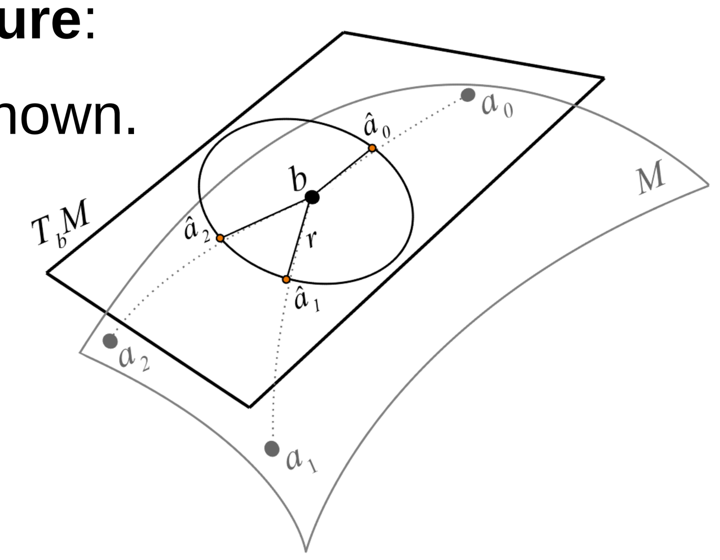
Optimal **solutions are acyclic** [2]:

⇒ **Edge flows** follow from flow conservation.

Solutions have **optimal substructure**:

⇒ Optimal branching angles are known.

⇒ Each branching points should have degree 3.

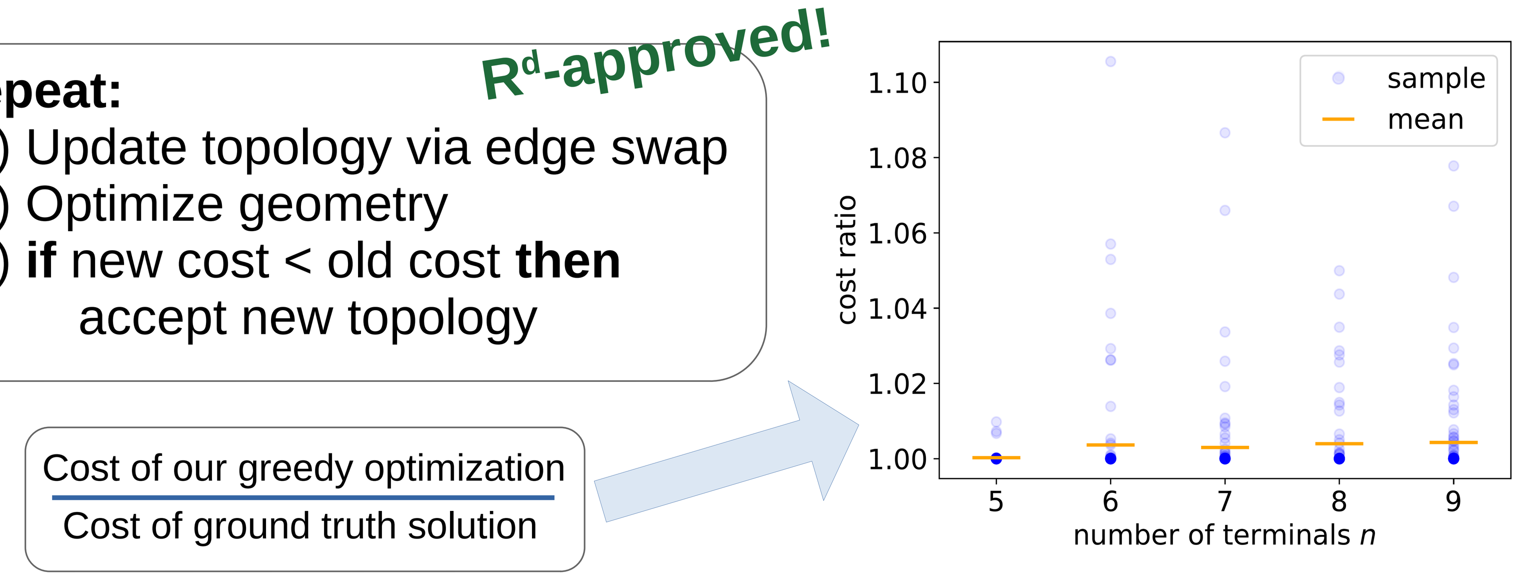


Generalizes also to manifolds!

Our greedy BOT optimization

Repeat:
1) Update topology via edge swap
2) Optimize geometry
3) **if** new cost < old cost **then** accept new topology

R^d-approved!



Geometry optimization

Given a topology, optimize geometry by iteratively solving:

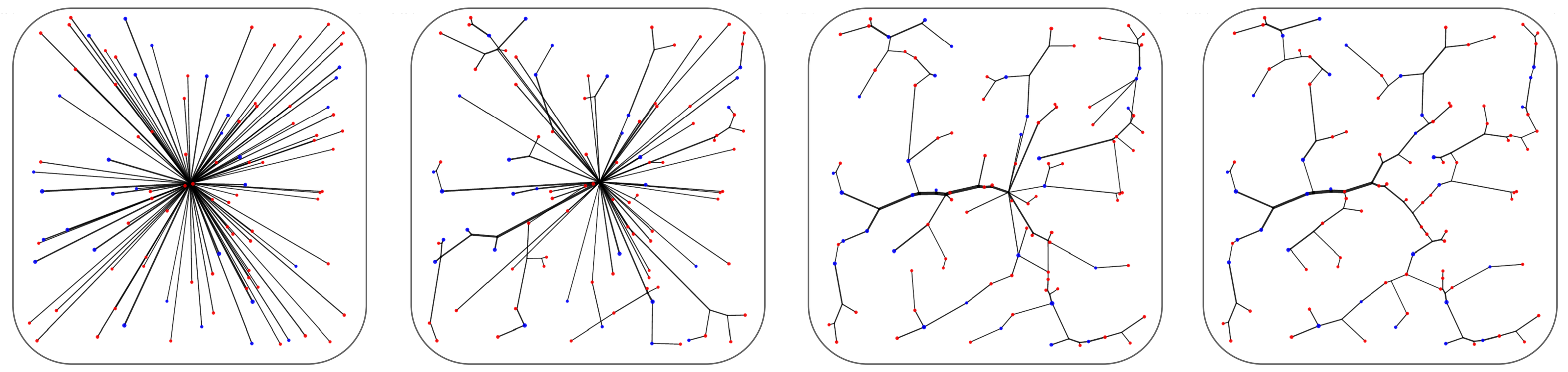
$$x_i^{(k+1)} = \sum_{j:(i,j)\in T} m_{ij}^{\alpha} \frac{x_j^{(k+1)}}{|x_i^{(k)} - x_j^{(k)}|} \Bigg/ \sum_{j:(i,j)\in T} \frac{m_{ij}^{\alpha}}{|x_i^{(k)} - x_j^{(k)}|}$$

Branching points — Edge flows

Each iteration can be solved in linear time [3].

Efficient C++ implementation available now!

Topology optimization



0 iterations. — 100 iterations. — 400 iterations. — 1977 iterations.

More examples and experiments at https://github.com/hci-unihd/BranchedOT.

References

[1] Gilbert, E. N. Minimum cost communication networks. Bell System Technical Journal, 46(9):2209–2227, 1967
[2] Bernot, M., Caselles, V., and Morel, J.-M. Optimal transportation networks: models and theory. Springer, 2008
[3] Smith, W. D. How to find steiner minimal trees in euclidean d-space. Algorithmica, 7(1):137–177, 1992