An embedding of sliced optimal transport metrics

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joint work with Asuka Takatsu[2]

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Outline

1. Background
2. An embedding

Monge-Kantorovich metrics

Definition (Monge-Kantorovich metrics)

For $1 \leq p < \infty$, μ, $\nu \in \mathcal{P}_p(\mathbb{R}^n)$ (finite pth moments)

$$\mathrm{MK}_p^{\mathbb{R}^n}(\mu,\nu) := \left(\inf_{\gamma \in \Pi(\mu,\nu)} \int_{\mathbb{R}^n \times \mathbb{R}^n} |x-y|^p d\gamma(x,y) \right)^{\frac{1}{p}},$$

$$\Pi(\mu,\nu) := \{\gamma \in \mathcal{P}(\mathbb{R}^n \times \mathbb{R}^n) \mid \text{left/right marginals} = \mu/\nu\}.$$

$\mathrm{MK}_p^{\mathbb{R}^n}$ is a metric on $\mathcal{P}_p(\mathbb{R}^n)$.

$\mathrm{MK}_p^{\mathbb{R}^n}$ is nice, but expensive to compute.

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Sliced distances

Idea: 1D OT is super easy to compute

Definition (Julien-Peyré-Delon-Bernot, 2011)

For $\omega \in \mathbb{S}^{n-1}$, def. $R^\omega : \mathbb{R}^n \to \mathbb{R}$ by $R^\omega(x) := \langle x, \omega \rangle$.
The sliced Wasserstein distance for μ, $\nu \in \mathcal{P}_p(\mathbb{R}^n)$ is

$$SW_p(\mu, \nu) := \left(\int_{\mathbb{S}^{n-1}} \mathrm{MK}_p^{\mathbb{R}}(R^\omega_\# \mu, R^\omega_\# \nu)^p d\sigma(\omega) \right)^{\frac{1}{p}},$$

σ is normalized surface measure.

Definition (Deshpande et. al., 2019)

The max-sliced Wasserstein distance for μ, $\nu \in \mathcal{P}_p(\mathbb{R}^n)$ is

$$MW_p(\mu, \nu) := \mathrm{ess\ sup}_\sigma \mathrm{MK}_p^{\mathbb{R}}(R^\bullet_\# \mu, R^\bullet_\# \nu).$$

For $\mu \in \mathcal{P}_p(\mathbb{R}^n)$, $R^\omega_\# \mu \in \mathcal{P}_p(\mathbb{R})$ is the Radon transform of μ.

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Known properties

Suggested idea: SW_p or MW_p which is computationally easier, so use in place of $\mathrm{MK}_p^{\mathbb{R}^n}$.

Known:

- SW_p, MW_p are metrics on $\mathcal{P}_p(\mathbb{R}^n)$
- $SW_p \lesssim \mathrm{MK}_p^{\mathbb{R}^n}$,

  $\mathrm{MK}_p^{\mathbb{R}^n} \lesssim SW_p^{\frac{1}{n+1}}$ for compactly supported (Bonnotte, thesis)
- $\mathrm{MK}_p^{\mathbb{R}^n}$, SW_p, MW_p topologically equiv.
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Slight generalization

Definition

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$\mathrm{MK}_{p,p} = SW_p$, $\mathrm{MK}_{p,\infty} = MW_p$.
Note: if $n = 1$, $\mathrm{MK}_{p,q} = \mathrm{MK}_p^{\mathbb{R}}$, so assume $n \geq 2$.

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Properties of sliced Monge-Kantorovich

Theorem (K.-Takatsu)

For $1 \leq p < \infty$, $1 \leq q \leq \infty$,

- $(\mathcal{P}_p(\mathbb{R}^n), \mathrm{MK}_{p,q})$ *is a complete, separable, metric space*
- $\mathrm{MK}_{p,q} \lessapprox \mathrm{MK}_p^{\mathbb{R}^n}$, $\mathrm{MK}_{p,q}(\mu, \delta_x) = C_{p,q}\, \mathrm{MK}_p^{R^n}(\mu, \delta_x)$ *any* μ, x.
- $(\mathcal{P}_p(\mathbb{R}^n), \mathrm{MK}_{p,q})$ *is not a geodesic space.*

$\gamma : [0,1] \to X$ is a geodesic in (X, d) if $d(\gamma(s), \gamma(t)) = |t-s| d(\gamma(0), \gamma(1))$ all s, $t \in [0,1]$.

(X, d) is a geodesic space if for every x_1, $x_2 \in X$, there is a geodesic connecting them.

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Nongeodesic example

$\mu_0 := \frac{1}{2}(\delta_{e_1} + \delta_{-e_1}),\ \mu_1 := \frac{1}{2}(\delta_{e_2} + \delta_{-e_2})$

If μ_t is $\mathbb{MK}_{p,q}$ geodesic, must have $R^\omega_\# \mu_t = \mathbb{MK}^{\mathbb{R}}_p$ geodesic from $R^\omega_\# \mu_0$ to $R^\omega_\# \mu_1$.

$$R^\omega_\sharp \mu_t = \begin{cases} \frac{\left(\delta_{\langle \omega, (1-t)e_1 + te_2 \rangle} + \delta_{-\langle \omega, (1-t)e_1 + te_2 \rangle}\right)}{2}, & |\langle \omega, e_1 - e_2 \rangle| < |\langle \omega, e_1 + e_2 \rangle|, \\ \frac{\left(\delta_{\langle \omega, (1-t)e_1 - te_2 \rangle} + \delta_{-\langle \omega, (1-t)e_1 - te_2 \rangle}\right)}{2}, & |\langle \omega, e_1 - e_2 \rangle| > |\langle \omega, e_1 + e_2 \rangle|. \end{cases}$$

Can show $\operatorname{spt} \mu_t \in (\{\pm(e_1 - e_2)\} \cap \{\pm(e_1 + e_2)\}) \times \mathbb{R}^{n-2} = \emptyset$, contradiction.

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What went wrong

We could not guarantee for each time that $\omega \mapsto \mathrm{MK}_p^{\mathbb{R}}$ geod. from $R^{\omega}_{\#}\mu_0$ to $R^{\omega}_{\#}\mu_1$ is in the image of Radon transform.

Idea: replace Radon transform by disintegration.

Theorem (Disintegration of measures)

Let $\mathfrak{m} \in \mathcal{P}^{\sigma}(\mathbb{R} \times \mathbb{S}^{n-1})$ ($\mathfrak{m} \in \mathcal{P}(\mathbb{R} \times \mathbb{S}^{n-1})$, $\mathbb{S}^{n-1}$ marginal is σ). Then for σ-a.e. ω, there is $\mathfrak{m}^{\omega} \in \mathcal{P}(\mathbb{R})$ s.t.

$$\omega \mapsto \mathfrak{m}^{\omega}(A) \text{ is Borel for any Borel } A \subset \mathbb{R},$$

$$\int_{\mathbb{R}\times\mathbb{S}^{n-1}} f d\mathfrak{m} = \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} f(t,\omega) d\mathfrak{m}^{\omega}(t) d\sigma(\omega), \text{ any Borel function } f.$$

Write $\mathfrak{m} = \mathfrak{m}^{\omega} \otimes d\sigma$

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Let $\mathfrak{m} \in \mathcal{P}^{\sigma}(\mathbb{R} \times \mathbb{S}^{n-1})$ *(*$\mathfrak{m} \in \mathcal{P}(\mathbb{R} \times \mathbb{S}^{n-1})$*,* $\mathbb{S}^{n-1}$ *marginal is* σ*). Then for* σ*-a.e.* ω*, there is* $\mathfrak{m}^{\omega} \in \mathcal{P}(\mathbb{R})$ *s.t.*

$$\omega \mapsto \mathfrak{m}^{\omega}(A) \text{ is Borel for any Borel } A \subset \mathbb{R},$$

$$\int_{\mathbb{R}\times\mathbb{S}^{n-1}} f d\mathfrak{m} = \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} f(t,\omega) d\mathfrak{m}^{\omega}(t) d\sigma(\omega), \text{ any Borel function } f.$$

Write $\mathfrak{m} = \mathfrak{m}^{\omega} \otimes d\sigma$

Disintegrated Monge-Kantorovich

Definition (K.-Takatsu)

Let $1 \leq p < \infty$, $1 \leq q \leq \infty$.
The disintegrated Monge-Kantorovich distance for $\mathfrak{m} = \mathfrak{m}^\omega \otimes d\sigma$, $\mathfrak{n} = \mathfrak{n}^\omega \otimes d\sigma \in \mathcal{P}^\sigma_{p,q}(\mathbb{R} \times \mathbb{S}^{n-1})$ is

$$\mathcal{MK}_{p,q} := \|\mathrm{MK}^{\mathbb{R}}_p(\mathfrak{m}^\bullet, \mathfrak{n}^\bullet)\|_{L^q(\sigma)},$$

where

$$\mathcal{P}^\sigma_{p,q}(\mathbb{R} \times \mathbb{S}^{n-1}) := \left\{ \mathfrak{m} \in \mathcal{P}^\sigma(\mathbb{R} \times \mathbb{S}^{n-1}) \mid \| \int_{\mathbb{R}} |t|^p d\mathfrak{m}^\bullet(t) \|_{L^{q/p}(\sigma)} \right\}$$

Can view this as $L^q(\sigma; (\mathcal{P}_p(\mathbb{R}), \mathrm{MK}^{\mathbb{R}}_p))$
(metric space valued L^q space)

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Properties of disintegrated Monge-Kantorovich

Theorem (K.-Takatsu)

For $1 \leq p < \infty$, $1 \leq q \leq \infty$,

- $(\mathcal{P}^{\sigma}_{p,q}(\mathbb{R} \times \mathbb{S}^{n-1}), \mathcal{MK}_{p,q})$ *is a complete, separable* ($q < \infty$)*, geodesic, metric space*
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$$d((t_1, \omega_1), (t_2, \omega_2)) = \begin{cases} |t_1 - t_2|, & \omega_1 = \omega_2, \\ \infty, & \text{else.} \end{cases}$$

- $\mu \mapsto R^{\omega}_{\#}\mu \otimes d\sigma$ *is an isometric embedding* $(\mathcal{P}_p(\mathbb{R}^n), \mathrm{MK}_{p,q}) \hookrightarrow (\mathcal{P}^{\sigma}_{p,q}(\mathbb{R} \times \mathbb{S}^{n-1}), \mathcal{MK}_{p,q})$

Comment: intrinsic distance on embedded $(\mathcal{P}_p(\mathbb{R}^n), \mathrm{MK}_{p,p})$ induced by $\mathcal{MK}_{p,p}$ is $\mathrm{MK}^{R^n}_{p}$ (Candau-Tilh, master's thesis, compact support case)

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Duality

We actually have a dual problem (holds for sliced too).

Theorem (K.-Takatsu)

$1 \leq p \leq q \leq \infty$. *Then*

$$\mathcal{MK}_{p,q}(\mathfrak{m}, \mathfrak{n})^p =$$
$$\sup \Big\{ -\int_{\mathbb{R}\times\mathbb{S}^{n-1}} \zeta\Phi d\mathfrak{m} - \int_{\mathbb{R}\times\mathbb{S}^{n-1}} \zeta\Psi d\mathfrak{n} \mid \zeta \in C_b(\mathbb{S}^{n-1}),$$
$$\|\zeta\|_{L^{(q/p)'}(\sigma)} \leq 1,\ \Phi, \Psi \in C_b(\mathbb{R}\times\mathbb{S}^{n-1}),$$
$$-\Phi(t,\omega) - \Psi(s,\omega) \leq |t-s|^p \Big\}$$

Barycenters

We actually have existence of barycenters (also sliced) and duality for barycenters à la Agueh-Carlier (not for sliced).

Theorem (K.-Takatsu)

$1 \leq p \leq q \leq \infty$, $\sum \lambda_i = 1$, $\lambda_i \geq 0$, $\mathfrak{m}_i \in \mathcal{P}^{\sigma}_{p,q}(\mathbb{R} \times \mathbb{S}^{n-1})$.

$$\min_{\mathfrak{n}} \sum_{i=1}^{N} \lambda_i \mathcal{MK}_{p,q}(\mathfrak{m}_i, \mathfrak{n})^p =$$

$$\sup \left\{ -\sum_{i=1}^{N} \int_{\mathbb{S}^{n-1}} \zeta_i \int_{\mathbb{R}} S_{\lambda_i,p} \xi_i^{\omega} d\mathfrak{m}_i^{\omega} d\sigma(\omega) \mid \sum_{i=1}^{N} \zeta_i \xi_i \equiv 0, \right.$$

$$\left. \zeta_i \in C_b(\mathbb{S}^{n-1}), \|\zeta_i\|_{L^{(q/p)'}(\sigma)} \leq 1, \frac{\xi_i}{1+|t|^p} \in C_0(\mathbb{R} \times \mathbb{S}^{n-1}) \right\}$$

where $S_{\lambda_i,p}\xi^{\omega}(s) := \sup_{t \in \mathbb{R}}(-\lambda_i |t-s|^p - \xi(t,\omega))$.

Thank you

Thank you very much!