Monge-Ampère gravitation as a Γ-limit of good rate functions

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The Vlasov-Monge-Ampère system

The VMA system in the space domain $\mathbb{T}^d$ writes for $(t, x, v) \in \mathbb{R}_+ \times \mathbb{T}^d \times \mathbb{R}^d$

$$\boxed{\begin{cases} \partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) - \nabla_x \varphi(t,x) \cdot \nabla_v f(t,x,v) = 0, \\ \det(\mathrm{Id} + D_x^2 \varphi(t,x)) = \displaystyle\int f(t,x,v)\, \mathrm{d}v, \\ f(t,x,v)|_{t=0} = f_0(x,v). \end{cases}}$$

where $f \geq 0$ and $\iint f(t,x,v)\, \mathrm{d}x\, \mathrm{d}v = 1$.

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where $f \geq 0$ and $\iint f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v = 1$. As

$$\det(\mathrm{Id} + D_x^2\varphi(t,x)) \approx 1 + \mathrm{tr}(D_x^2\varphi(t,x)) = 1 + \Delta_x \varphi(t,x),$$

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Zeldovich approximation

In the context of the evolution of the density of matter in an Einstein-de Sitter Universe, this change from the Poisson equation to the Monge-Ampère equation is an approximation for which the Zeldovich approximation becomes exact.

$$\begin{cases} \partial_t v + (v \cdot \nabla) v = -\dfrac{3}{2t}(v + \nabla\varphi), \\ \partial_t \rho + \mathrm{div}(\rho v) = 0, \\ 1 + t\Delta\varphi = \rho. \end{cases}$$

"Reconstruction of the early Universe"

[Frisch, Matarrese, Mohayaee, Sobolevskii 2002]

[Brenier, Frisch, Hénon, Loeper, Matarrese, Mohayaee, Sobolevskii 2003]

An optimal transport interpretation

The characteristics of the VMA system are the solutions of the ODE

$$\ddot{X}_t = -\nabla_x \varphi(t, X_t),$$

where $T_t : x \mapsto x + \nabla_x \varphi(t, x)$ is the *optimal map* in the quadratic OT problem sending $\rho(t, \cdot) = \int f(t, \cdot, v)\, \mathrm{d}v$ onto the Lebesgue measure.

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Therefore, the whole system can be reformulated as follows:

$$\begin{cases} \ddot{X}_t(x, v) = X_t(x, v) - T_t(X_t(x, v)), \\ X_0(x, v) = x, \quad \dot{X}_0(x, v) = v, \\ T_t \text{ sends optimally } X_{t\#} f_0 \text{ onto } \mathrm{Leb}. \end{cases}$$

(The phase-space density f is given from X by $f(t, \cdot) = (X_t, \dot{X}_t)_{\#} f_0$.)

A discrete version

From now on, we work on $\mathbb{R}^d$, and we replace Leb by and empirical measure

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{a_i}, \qquad a_1, \ldots, a_N \in \mathbb{R}^d.$$

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We reformulate this as:

$$X_t = (x_1(t),\ldots,x_N(t)) \quad \text{and for all } \sigma \in \mathfrak{S}_N, \quad A^{\sigma} = (a_{\sigma(1)},\ldots,a_{\sigma(N)}),$$

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This is well defined only when the Monge problem has a unique sol.

Dealing with singularities: looking for convexity

For $X = (x_1, \dots, x_N) \in (\mathbb{R}^d)^N$, let us call

$$F(X) := -\frac{N}{2} W_2^2 \left(\frac{1}{N} \sum_{i=1}^{N} \delta_{a_i}, \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \right) = -\min_{\sigma \in \mathfrak{S}_N} \frac{\left| X - A^\sigma \right|^2}{2},$$

$$f(X) := F(X) + \frac{1}{2}(|X|^2 + |A|^2) = \max_{\sigma \in \mathfrak{S}_N} A^\sigma \cdot X.$$

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At a point X where the Monge problem has a unique solution given by σ^*,

$$\nabla F(X) = -(X - A^{\sigma^*}), \qquad \nabla f(X) = A^{\sigma^*}.$$

and our discrete model writes formally

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But f is convex and F is -1-convex! [Ambrosio, Gangbo '08] restated MAG as

$$\ddot{X}_t \in -\partial F(X_t) \quad \text{or} \quad X_t - \ddot{X}_t \in \partial f(X_t).$$

Where our derivation leads

Our approach leads to a description of the system in the form of a least action principle. Due to our formal ODE, one could expect as an action

$$\int_{t_0}^{t_1} \left\{ \frac{1}{2} |\dot{X}_t|^2 - F(X_t) \right\} \mathrm{d}t.$$

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Instead, it selects the following very close one

$$\int_{t_0}^{t_1} \left\{ \frac{1}{2}|\dot{X}_t|^2 + \frac{1}{2}|X_t - \overline{\nabla} f(X_t)|^2 \right\} \mathrm{d}t,$$

where $\overline{\nabla} f(X)$ is the vector of smallest length in $\partial f(X)$.

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But for "bad" points, $\frac{1}{2}|X - \overline{\nabla} f(X)|^2 < -F(X)$, so they are favoured: when $d = 1$, the model presents sticky collisions! (See [Brenier '10].)

Our derivation: perturbation of a "pilot wave" ODE

We start with the density $\rho^\varepsilon(t,\cdot)$ in $(\mathbb{R}^d)^N$ at time $t>0$ of N indistinguishable Brownian particles of diff $\varepsilon>0$, initially in $a_1,\dots,a_N$ up to permutation:

$$\rho^\varepsilon(t,X)=\frac{1}{N!\sqrt{2\pi\varepsilon t}^{Nd}}\sum_{\sigma\in\mathfrak{S}_N}\exp\left(-\frac{|X-A^\sigma|^2}{2\varepsilon t}\right).$$

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Following the idea of pilot waves by Louis de Broglie in the context of quantum mechanics, we see this density as the solution of the continuity equation

$$\partial_t \rho^\varepsilon + \mathrm{div}(\rho^\varepsilon v^\varepsilon) = 0,$$

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$$=\frac{1}{2t}\left(X-\nabla f_\varepsilon(t,X)\right),\quad\text{where } f_\varepsilon(t,x):=\varepsilon t\log\sum_{\sigma\in\mathfrak{S}_N}\exp\left(\frac{A^\sigma\cdot X}{\varepsilon t}\right).$$

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Now, the idea is to perturb the characteristic ODE of this continuity equation as

$$\mathrm{d}X_t^{\varepsilon,\eta} = v^{\varepsilon}(t, X_t^{\varepsilon,\eta})\,\mathrm{d}t + \eta\alpha(t)\,\mathrm{d}B_t,$$

being (B_t) a standard Brownian motion and α a smooth function.

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Now, the spirit of the result is as follows:

> *When α is well chosen, $\eta \to 0$ and then $\varepsilon \to 0$, up to a change of time, the solutions of this SDE starting from P at time $t_0 > 0$ and which happen to be close to Q for a further time $t_1 > t_0$ are close to minimizers the action previously described.*

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From now on, we fix $0 < t_0 < t_1$ and $P \in (\mathbb{R}^d)^N$, and we call $\mu_{\varepsilon,\eta}$ the law of our SDE starting from P at time t_0, up to time t_1. Finally, we choose a final point $Q \in (\mathbb{R}^d)^N$.

Limit $\eta \to 0$: Freidlin-Wentzell large deviation principle

Theorem

For fixed $\varepsilon > 0$ and $\eta \to 0$, the family of laws $(\mu_{\varepsilon,\eta})$ satisfies the LDP on $C^0([t_0, t_1]; (\mathbb{R}^d)^N)$ with good rate function defined for all $\mathcal{X} = (X_t)_{t \in [t_0, t_1]}$ by:

$$
\begin{cases}
\dfrac{1}{2} \displaystyle\int_{t_0}^{t_1} \dfrac{1}{\alpha(t)^2} |\dot{X}_t - v_\varepsilon(t, X_t)|^2 \, \mathrm{d}t, & \text{if } \mathcal{X} \in H^1([t_0, t_1]; (\mathbb{R}^d)^N) \\
 & \text{and } X_{t_0} = P, \\
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In particular, the family of conditioned laws $(\mu_{\varepsilon,\eta}(\cdot|X_{t_1} = Q))$ admit limit points, and these limit points only charge minimizers of the action

$$L_\varepsilon : \mathcal{X} \mapsto \begin{cases} \dfrac{1}{2}\displaystyle\int_{t_0}^{t_1} \frac{1}{\alpha(t)^2}|\dot{X}_t - v_\varepsilon(t, X_t)|^2 \, \mathrm{d}t, & \text{if } \mathcal{X} \in H^1([t_0, t_1]; (\mathbb{R}^d)^N) \\ & X_{t_0} = P \text{ and } X_{t_1} = Q, \\ +\infty, & \text{else.} \end{cases}$$

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As $\varepsilon \to 0$, we have the family of actions (L_ε) Γ-converges towards

$$L : \mathcal{X} \mapsto \begin{cases} \dfrac{1}{2}\displaystyle\int_{t_0}^{t_1} \dfrac{1}{\alpha(t)^2} \left| \dot{X}_t - \dfrac{X_t - \overline{\nabla} f(X_t)}{2t} \right|^2 \mathrm{d}t, & \text{if } \mathcal{X} \in H^1([t_0, t_1]; (\mathbb{R}^d)^N), \\ & X_{t_0} = P \text{ and } X_{t_1} = Q, \\ +\infty, & \text{else.} \end{cases}$$

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"Proof".

We have

$$v^\varepsilon(t,X) = \frac{1}{2t}\left(X - \nabla f_\varepsilon(t,X)\right) \quad \text{where} \quad f_\varepsilon(t,X) = \varepsilon t \log \sum_{\sigma \in \mathfrak{S}_N} \exp\left(\frac{A^\sigma \cdot X}{\varepsilon t}\right).$$

But for all $t > 0$,

$$f_\varepsilon(t,X) \underset{\varepsilon \to 0}{\longrightarrow} f(X).$$

See the last slide for the reason we find the extended gradient $\overline{\nabla}$. □

Change of time, choice of α and conclusion

Given an $\mathcal{X}$, we define $\mathcal{Y} : \theta \mapsto Y_\theta := X_{\exp(2\theta)}$, $\theta_0 := \frac{\log t_0}{2}$ and $\theta_1 := \frac{\log t_1}{2}$. We find that

$$L(\mathcal{X}) = \frac{1}{2} \int_{\theta_0}^{\theta_1} \frac{1}{2 \exp(2\theta) \alpha(\exp(2\theta))^2} \left| \dot{Y}_\theta - (Y_\theta - \overline{\nabla} f(Y_\theta)) \right|^2 \mathrm{d}\theta.$$

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Therefore, choosing $\alpha(t) := \frac{1}{\sqrt{2t}}$, we find that $\mathcal{X}$ minimizes L if and only if $\mathcal{Y}$ minimizes

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$$\frac{1}{2} \int_{\theta_0}^{\theta_1} |\dot{Y}_\theta - (Y_\theta - \overline{\nabla} f(Y_\theta))|^2 \, \mathrm{d}\theta.$$

But this action is equal to

$$\int_{\theta_0}^{\theta_1} \frac{|\dot{Y}_\theta|^2}{2} + \frac{|Y_\theta - \overline{\nabla} f(Y_\theta))|^2}{2} \, \mathrm{d}\theta.$$

up to a term only depending on the endpoints P and Q, so we are done.

Why the extended gradient?

Up to some manipulation, we use this result, stated here in a Hilbert space.

Theorem (Ambrosio, Baradat, Brenier '21)

Let H be a Hilbert space. If (f_n) is a family of uniformly λ-convex function on H, $\lambda \in \mathbb{R}$, and if $f_n \to f$ in the sense of Mosco convergence. If finally

$$\sup_n |\overline{\nabla} f_n(x_i)| < +\infty, \qquad i = 0, 1.$$

Then, calling EP the endpoint constraints corresponding to $x_0, x_1 \in H$,

$$\frac{1}{2}\int_0^1 \left\{ |\dot{X}_t|^2 + |\overline{\nabla} f_n(X_t)|^2 \right\} \mathrm{d}t + \mathrm{EP} \xrightarrow[n \to +\infty]{\Gamma} \frac{1}{2}\int_0^1 \left\{ |\dot{X}_t|^2 + |\overline{\nabla} f(X_t)|^2 \right\} \mathrm{d}t + \mathrm{EP}$$

in the topology of $C^0([0,1]; H)$.

See also [Monsaingeon, Tamanini, Vorotnikov '20] for similar results in the Wasserstein space.

Why the extended gradient?

Up to some manipulation, we use this result, stated here in a Hilbert space.

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Thank you!